ON COMPONENT-WISE REGULAR EXPANSIONS OF NON-SINGULAR MATRICES

P. ALLEN AND M. B. M. ELGINDI

Abstract. Given a nonsingular point matrix, we present an algorithm for contructing a regular interval matrix containing the given point matrix. The algorithm used is based upon a theorem which provides a criterion for checking regularity of interval matrices.

Key words: interval matrix, regular calculations

1. Introduction

An interval matrix is a matrix whose elements are intervals of the real line instead of points. Common ways to represent interval matrices are min-max form,

$$A^I = [B, C], C \geq B, \tag{1}$$

where B and C consist of the left and right endpoints of the intervals in A^I, respectively, and midpoint- radius form,

$$A^{\mathrm{I}} = A_{\mathrm{c}} \pm \Delta, \tag{2}$$

where A_c contains the centers of the intervals in A^I and Δ is a matrix consisting of the radii of the intervals in A^I.

Let A^I be an n×n interval matrix. A^I is said to be regular if and only if all point matrices in A^I are invertible, where a point matrix is a matrix of real numbers.

Definition: Radius of Regularity Given matrix A and positive matrix Δ, the radius of regularity, $\sigma(A,\Delta)$, is the smallest σ such that $A_c \pm \sigma\, \Delta$ is singular. If no such σ exists, the radius of regularity is defined to be ∞

Our interest is in finding efficient numerical methods to solve for the radius of regularity, given A and Δ.

Regularity of interval matrices is closely related to stability. Given a matrix $A^I = A_c \pm \Delta$, if A_c is stable, and A^I is regular, A^I is also stable [1]. Stability is critical for many real-world applications, and knowing how far a system is, componentwise, from instability is useful for such applications. More information regarding stability and regularity of interval matrices can be found in Rohn [1].

2. Regular Norm-Wise Expansion

Using the norm-wise distance between two matrices, we can expand a non- singular matrix to a regular interval matrix. Our expansion is based on a well-known theorem, see Rudin [2], for example.

Theorem 1. *Given an invertible matrix, A, suppose*

$$\| A - B \| \| A^{-1} \| < 1. \tag{3}$$

Then, B is invertible.

Proof: Let $\beta = \| A - B \|$ and $\alpha = 1 / \| A^{-1} \|$.

$$\alpha \mid x \mid = \alpha \mid AA^{-1}x \mid \leq \alpha \| A^{-1} \| \| Ax \mid = \mid (A - B + B)x \mid \leq \mid (A - B)x \mid + \mid Bx \mid \tag{4}$$

$$0 \leq (\alpha - \beta) \mid x \mid \leq \mid Bx \mid, \tag{5}$$

since $\alpha > \beta$. The last inequality implies that the null space of B has dimension 0, and so B is invertible. □

Theorem 2. *Given an invertible matrix, $A_1 \in A^I = [B, C]$, the expansion given by:*

$$\begin{gathered} D^I = (B_1, C_1), where \\ B_1 = A_1 - \frac{1}{\|A_1 - B\| \|A_1^{-1}\|}(A_1 - B) \\ and \\ C_1 = A_C + \frac{1}{\|A_C - B\| \|A_1^{-1}\|}(A_1 - B) \end{gathered}$$

is regular.

Proof: Suppose $A = B_1$ or $A = C_1$. Then,

$$\| A - A_1 \| = \frac{1}{\| A_C - B \| \| A_1^{-1} \|} \| (A_C - B) \| = \frac{1}{\| A_1^{-1} \|} \tag{6}$$

So, since the endpoints B$_1$ and C$_1$ are not included in D^I, $then for any point matrix, E, in D^I$,

$$\| A_{Cij} - E_{ij} \| < \| A_{Cij} - B_{1ij} \| \Rightarrow \| A_C - E \| < \| A_C - B_1 \| = \frac{1}{\| A_1^{-1} \|} \tag{7}$$

So, by theorem 1, all point matrices in D^I are invertible, and thus D^I is regular. □

This expansion yields a computationally cheap method to regularly expand around a nonsingular matrix. This expansion can also be done replacing B in equation (6) with C.

3. Interval Determinant of an Interval Matrix

In this section we present a theorem which tells us how to calculate the range of the determinant over an interval matrix. Not only will this give us information about the regularity of an interval matrix, but it will give us a clue about how regular it is. In the next section we use this to construct an efficient numerical method for finding the radius of regularity.

Unfortunately, we cannot simply calculate the determinant of an interval matrix using interval arithmetic because the range of an interval function and its interval evaluation differ. A simple example is given by the function $f(x) = x^2$. If we let $x = [-1, 1]$, the range of the function f(x) is [0,1]. However, its interval evaluation, [-1,1]*[-1,1], is [-1,1]. The range of a function will always be a subset of its interval evaluation. So, we could use the interval evaluation of determinant as a sufficient condition for regularity, if the range of values is strictly positive or strictly negative. Alefeld and Herzberger [3] go into more detail outlining relations between the interval evaluation and the range of a function.

In general, there is no efficient way to calculate the range of an interval function. To calculate the range of an interval function, we must use our knowledge about that function. By using our knowledge of the determinant function, we will show that the determinant of an interval matrix can be calculated simply in accordance with the following theorem.

Theorem 3. *Let $A^I = [B, C]$ be an interval matrix, and let Γ be the set of all matrices, A, in A^I such that*

$$A_{ij} = B_{ij} \text{ or } C_{ij}. \tag{8}$$

Then, the range of the determinant over A^I is [Min(DetΓ),Max(DetΓ)].

Proof: Since determinant over $\mathrm{A^I}$ is a continuous function of n^2 variables over a compact region, it must attain a minimum and maximum values over that region.

Suppose the maximum value of the determinant is obtained at a matrix $A \in \mathrm{A^I}$ and that neither this value, nor a greater value, is obtained at a matrix in Γ. Let k be the number of entries, $\mathrm{a_{ij}}$, of A not satisfying (9). Let us call these entries $\mathrm{a_1}$ through $\mathrm{a_k}$. Beginning with n=1, we can expand the determinant of A in the following way:

$$Det(A) = a_{\mathrm{n}} * b + c \tag{9}$$

Where b is the quantity $\mathrm{a_n}$ is multiplied by in this expansion of the determinant, and c is all other values going into the evaluation of the determinant of A. Now, if b is negative, we take $\mathrm{a_n} = \mathrm{B_{i(n)j(n)}}$, where i(n) and j(n) are the indices of $\mathrm{a_n}$. If b is positive, we take $\mathrm{a_n} = \mathrm{C_{i(n)j(n)}}$. Either of these cases yields a matrix in $\mathrm{A^I}$ with determinant greater than the determinant of A, giving a contradicton.

If b is zero, let $\mathrm{a_n} = \mathrm{C_{i(n)j(n)}}$ Continue with this process for all k entries, or until a contradiction arises.

This process leads to a matrix in Γ. The determinant of this matrix is equal to the determinant of A, contradicting our assumption, proving that the maximum value of $\mathrm{Det(A^I)}$ occurs at an element in Γ.

A completely analagous process shows the minimum value for the determinant of a matrix in A occurs at an element in Γ. $\square$

This result allows us to check regularity of an interval matrix by checking if its range contains zero. In [4], Hansen proved a similar result, concluding that a matrix contains both negative and positive values for its determinant iff the matrices in Γ, as defined above, attain both negative and positive values for their determinants.

4. Algorithms for Regular Expansion

For an n×n interval matrix $\mathrm{A^I}$, calculating the determinant of all matrices in Γ requires 2^{n^2} determinant calculations. Therefore, although this method is simple, it requires many computations. We develop two similar algorithms which limit the number of times we are required to compute the interval matrix determinant.

Algorithm 1

Given a nonsingular matrix A, and a non-zero matrix Δ, this algorithm finds σ(A,Δ).

1. Let σ_0 be 0, and let F(σ_0) be the determinant of A
2. Let σ_1 be $\dfrac{1}{\rho(|A|\Delta)}$, a lower bound for σ(A,Δ). Let F(σ_1) be the minimum value in the determinant of A±σ_1Δ if F(0) is positive, and let F(σ_1) be the maximum of that determinant if F(0) is negative
3. Let $\sigma_2 = \sigma_1 - \dfrac{F(\sigma_1)}{\frac{F(\sigma_1)-F(\sigma_0)}{\sigma_1-\sigma_0}}$ (Secant Method)
4. Repeat step (3) increasing each subscript by 1, until F(σ_n) has absolute value below a threshold value.

Algorithm 2

Given a nonsingular matrix A, and a non-zero matrix Δ, this algorithm finds σ(A,Δ).

1. Let σ_0 be 0, and let F(σ_0) be the determinant of A
2. Let $\sigma_1 = \dfrac{Min\Delta_{i,j}}{Max\Delta_{i,j}^2}$. Let F($\sigma_1$) be the minimum value in the determinant of A±σ_1Δ if F(0) is positive, and let F(σ_1) be the maximum of that determinant if F(0) is negative
3. Let $\sigma_2 = \sigma_1 - \dfrac{F(\sigma_1)}{\frac{F(\sigma_1)-F(\sigma_0)}{\sigma_1-\sigma_0}}$ (Secant Method)
4. Repeat step (3) increasing each subscript by 1, until F(σ_n) has absolute value below a threshold value.

These algorithms both use the secant method to approximate the zero for F(σ). The former uses a lower bound for radius of regularity described in [5], whereas the latter uses a lower bound for radius of regularity based on theorem 2.

The initial guess used in algorithm 1 has been shown to be strictly better than other norm-based lower bounds similar to the one used in algorithm 2 in [5]. However, the initial guess used in algorithm 2 is computationally cheaper. There may be instances where algorithm 2 is preferable to algorithm 1 for this reason.

5. Examples

Given $A = \begin{pmatrix} 5 & 3 & -2 & 7 \\ 1 & -2 & -5 & 8 \\ 12 & 3.5 & 7 & -5.7 \\ 1.2 & 10 & -3 & 4 \end{pmatrix}$ With determinant: 1680.24

And $\Delta = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$

Using algorithm 1, we expand to:

$$\begin{pmatrix} [4.7067, 5.2933] & [2.7067, 3.2933] & [-2.2933, -1.7067] & [6.7067, 7.2933] \\ [0.7067, 1.2933] & [-2.2933, -1.7067] & [-5.2933, -4.7067] & [7.7067, 8.2933] \\ [11.7067, 12.2933] & [3.2067, 3.7933] & [6.7067, 7.2933] & [-5.9933, -5.4067] \\ [0.9067, 1.4933] & [9.7067, 10.2933] & [-3.2933, -2.7067] & [3.7067, 4.2933] \end{pmatrix}$$

With determinant: [0.0000, 3518], rounded. So, σ(A,Δ) was found to be .2933.

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P. Allen, University of Maryland Baltimore County, Baltimore, MD, 21250
E-mail address: `allenp1@umbc.edu`

M. B. M. Elgindi, Department of Mathematics, University of Wisconsin, Eau Claire, WI 54702-4004
E-mail address: `elgindmb@uwec.edu`