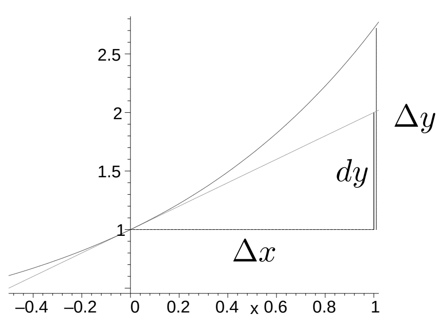
Lesson 18
Section 2.9
Differentials; Linear and Quadratic Approximations

We regard the symbol $\dfrac{dy}{dx}$ as a single value and not a ratio of two values. Now we wish to define dy and dx as separate quantities. We do so with the following definition:

> **Definition:** Let $y = f(x)$ be a differentiable function. Define dx to be an independent variable taking on any real value. Define dy as a dependent variable depending on both x and dx by the following equation:
>
> $$dy = f'(x) \cdot dx.$$

Geometrically dy is the change of the y-value of the tangent line of $y = f(x)$ as x changes by dx.

Recall that $\Delta y = f(x + \Delta x) - f(x)$ is the change of the y value of $y = f(x)$ as x changes by Δx. Since dx is an independent variable we can choose $dx = \Delta x$. Thus $dy = f'(x) \cdot \Delta x$. For small values of Δx the tangent line to $y = f(x)$ is approximately the same as $y = f(x)$. Thus dy is an approximately Δy.



To understand the value of differentials you have to imagine how the values of trigonometric functions are computed. There are certain values for which we know the value of the Trigonometric functions but other values are not known. Differentials allow us the approximate the values of the Trigonometric functions close to the known values.

Example: Find the value of $\sin(0.1)$.

We have for any function $f(a + \Delta x) = f(a) + \Delta y$. If the function is differentiable we we may approximate by $f(a + \Delta x) \approx f(a) + dy$.

For $f(x) = \sin(x)$, we have $f'(x) = \cos(x)$. The differential is $dy = f'(x) \cdot dx$. Letting $a = 0$ and $dx = \delta x = 0.1$ we have $dy = \cos(0) \cdot 0.1 = 1 \cdot 0.1 = 0.1$. Thus $\sin(0.1) \approx \sin(0) + 0.1 = 0 + 0.1 = 0.1$

In this example the sine function was approximated by its tangent line. We say the tangent line is the **linear approximation** of a differentiable function at a point a. The equation of the tangent line is $y = f(a) + f'(x) \cdot (x - a)$, thus the linear approximation is the function

$$L(x) = f(a) + f'(x) \cdot (x - a).$$

A better approximation than the linear approximation is the **quadratic approximation**. The quadratic approximation is a quadratic function that agrees with the original function at a point a, that has the same tangent line as the original function. The equation of the quadratic approximation is given by

$$Q(x) = f(a) + f'(a) \cdot (x - a) + \frac{1}{2} \cdot f''(a) \cdot (x - a)^2.$$

Problems

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