APPENDIX 1

The Continuum Hypothesis

A major question arising from the very beginning of Set Theory is one regarding the relationship between ordinal numbers and cardinal numbers. Namely, which ordinal is the first uncountable ordinal? We know that ordinal numbers and cardinal numbers are identical until the ordinal $\omega+1$ which has the same cardinality as ω, namely $\aleph_0$. Since we know that $2^{\aleph_0}$ has cardinality greater than $\aleph_0$ could that be the next cardinal number? I.e. does there not exist an ordinal number whose cardinality is strictly greater than $\aleph_0$ and strictly less than $2^{\aleph_0}$? We call $\aleph_1$ the next cardinal greater than $\aleph_0$, and we formally state the hypothesis $\aleph_1 = 2^{\aleph_0}$, which is known as the continuum hypothesis.

Recall that cardinal numbers are ordinal numbers so the existence of the next larger cardinal number is guaranteed by the well ordering of ordinal numbers. If we let $\aleph$ be an arbitrary cardinal number we designate the next larger cardinal number by $\aleph_+$. We can thus generalize the continuum hypothesis by

$$\aleph_+ = 2^{\aleph} \ \forall \aleph \geq \aleph_0.$$

Paul Cohen demonstrated in 1963 that the answer to the above question cannot be decided with the Zermelo-Fraenkel axioms. That is the continuum hypothesis is independent of ZF Axioms! A sketch of the argument that establishes this fact is given in Keith Devlin's book *The Joy of Sets*, and a rigorous account is given in Bell's book *Boolean-Valued Models and Independence Proofs in Set Theory.*