XI SURREAL NUMBERS

The origin of Surreal numbers is credited to John Conway, however the name was coined by Don Knuth. For consistency we may use the symbol $\mathbb{S}$ for surreal numbers, however the need to do so seldom arises.

The Constructive Definition of Surreal Numbers

John Conway defined Surreal numbers by formulating two simple rules for the construction of Surreal numbers plus the definitions for addition and multiplication. With these two rules and two definitions, a collection of numbers is constructed that includes all Real numbers, and all Ordinal numbers. The collection although not a set, but rather a proper class, forms a field.

Rule 1: Every number is represented by a pair of sets of previously constructed numbers, a left set and a right set, where no number in the left set is greater than or equal to any number of the right set.

Rule 2: A number, a, is less than or equal to a number, b, if and only if no member of $a's$ left set is greater than or equal to b, and no member of $b's$ right set is less than or equal to a.

The first rule tells us how to construct new Surreal numbers from previously constructed numbers. The second rule defines the order relation of the collection of surreal numbers that is necessary for the construction.

We will develop the surreal numbers using an alternate definition, that uses some of the principles already developed in Set Theory.

The Function Definition of Surreal Numbers

Definition A **Surreal number** is a function from an ordinal number to a two point space. The two point space is designated by $\{+, -\}$. The domain of a surreal number will be called its **length**.

The function from the ordinal number $0 = \{\}$ is of course the empty function,$\{\}$, and is considered to be a surreal number, and we call it 0.

Example $5 = \{0, 1, 2, 3, 4\}$. Thus

$$\begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ + & + & - & + & - \end{array}$$

Is a surreal number

Recall that we defined a sequence as a function whose domain is an ordinal number. If the domain of a sequence is the ordinal number α then we say that the sequence is an α-sequence.

Thus a surreal number, a, is a binary valued α-sequence for some ordinal α, and its length is α, which we indicate by $l(a) = \alpha$. The 0-sequence is of course the empty set.

Definition If a is an α-sequence, and b is a β-sequence, such that $a \cap b = b$, then b is an **initial segment** of a. If $b \neq a$, then b is a **proper initial segment**. We see that if b is a proper initial segment of a, then $l(b) < l(a)$.

Two surreal numbers are equal if they are equal as sets.

We define a linear order on the class of surreal numbers by the following:

Let a and b be surreal numbers, and let c be the maximal initial segment that is in both a and b, where c is a γ-sequence. We say

$$a > b \text{ if } \begin{cases} a(\gamma) = + & \text{and } b(\gamma) = - \text{ or} \\ a(\gamma) = + & \text{and } b(\gamma) \text{ is undefined or} \\ a(\gamma) \text{ is undefined} & \text{and } b(\gamma) = - \text{ .} \end{cases}$$

The Canonical Representation of Surreal Numbers

Let a be a surreal number. If $A' = \{a' | a'$ is an initial segment of a and $a' < a\}$, and $A'' = \{a'' | a''$ is an initial segment of a and $a < a''\}$, then we say $A'|A''$ is the **canonical representation** of a.

Example $(++--+) = 1\frac{3}{8} = \{0, 1, 1\frac{1}{4}\}|\{2, 1\frac{1}{2}\}$

Addition of Surreal Numbers

We define addition in the following way.

If $a = A'|A''$ and $b = B'|B''$ are in canonical form, then

$$a + b = \{a + b', b + a'\}|\{a + b'', b + a''\}$$

$$\forall\ a' \in A,'\ b' \in B',\ a'' \in A'',\ b'' \in B''.$$

We now verify that the elements of the left set are truly less than the elements of the right set. But first we must verify that addition is commutative.

$$a + b = \{a + b', b + a'\}|\{a + b'', b + a''\} = \{b + a', a + b'\}|\{b + a'', a + b''\} = b + a$$

To complete the verification we induct on the ordinal sum of the lengths of a and b. We make the inductive hypothesis that if $l(c) + l(d) < l(a) + l(b)$, $l(c') + l(d') < l(a) + l(b)$, $c < c'$, and $d' < d'$, then $c + d < c' + d'$.

We have $a' < a < a''$, $b' < b < b''$. Thus

$$a + b' < a + b'',\ a + b' < a'' + b = b + a''$$

and

$$b + a' < b + a'',\ b + a'' < b'' + a = a + b''.$$

Thus $a + b$ does represent a Surreal number. However we notice that $a + b$ is **not** in canonical form. Is it well defined?

If A and B are sets of surreal numbers such that $a < b\ \forall a \in A$ and $b \in B$, then $A|B$ is the "first" surreal number c, such that $a < c < b\ \forall a \in A$ and $b \in B$. That is, c is the surreal number with the minimal domain such that $a < c < b\ \forall a$ and $\forall b$. Hence the sum is well defined.

The way to think of why this is true is this way. Start at 0 and try to get into the gap between the two sets as quickly as possible. If there is an element of the lower set greater or equal to zero step towards the upper set, if the opposite is true step toward the lower set. If you arrive between the two sets then you have defined that "youngest" surreal number. If you are still "amid" one of the two sets, step towards the other set and continue until you arrive between them. If at any stage you were to step away from the set in which you are not amid, then you can never arrive between the sets, thus there is only one way to arrive between the sets, and when you first get there you stop.

Example Compute 1+1.

To add 1+1 we need to know 0+1, to add 0+1 we need to know 0+0.

We have $0 = \emptyset|\emptyset$, thus $0 + 0 = \{0 + b', 0 + a'\}|\{0 + b'', 0 + a''\}$, where $a', a'', b', b'' \in \emptyset \ \not\rightarrow$. Since there are no elements in the empty set to add to

0 each of those sums does not exist and thus

$$0 + 0 = \emptyset|\emptyset = 0.$$

We now have $1 = \{0\}|\emptyset$, thus

$$0 + 1 = \{0 + 0, 1 + a'\}|\{0 + b'', 1 + a''\} \text{ where } a', a'', b'' \in \emptyset \;\not\longrightarrow .$$

Thus we have $0 + 1 = \{0\}|\emptyset = 1$. Finally

$$1 + 1 = \{1 + 0, 1 + 0\}|\{1 + b'', 1 + a''\} \text{ where } a'', b'' \in \emptyset \;\not\longrightarrow .$$

Thus $1 + 1 = \{1\}|\emptyset = 2$.

We can see that $0 = \emptyset|\emptyset$ is the additive identity, since if we let $a = A'|A''$, then

$$a + 0 = \{0 + a'\}|\{0 + a''\} = a.$$

Now we can define the additive inverses of surreal numbers.

Let a be a surreal number. Define $-a$ as:

$$-a(\alpha) = \begin{cases} + & \text{if } a(\alpha) = - \\ - & \text{if } a(\alpha) = +. \end{cases}$$

Example Let $a + (+ - +) = \frac{3}{4} = \{0, \frac{1}{2}|\{1\}$,
thus $-a = (- + -) = -\frac{3}{4} = \{-1\}|\{0, -\frac{1}{2}\}$.

$$a + b = \{-\frac{1}{4}, -\frac{3}{4}\}|\{\frac{3}{4}, \frac{1}{4}\} = 0.$$

The canonical representation of the opposite of a given surreal number $a = A'|A''$ has the opposites of A'' as its lower set and the opposites of A'

as its upper set. Thus if $a = A'|A''$, then $-a = -A''| - A'$ where $-A'' = \{-a''|a'' \in A''\}$ and $-A' = \{-a'|a' \in A'\}$.

Now $a + (-a) = \{a + (-a''), (-a) + a'\}|\{a + (-a'), -a + a''\}$

Since $a' < a < a'' \Rightarrow -a'' < -a < -a'$ We have

$$\begin{aligned} a + (-a'') &< a'' + (-a'') = 0 \\ -a + (a') &< a' + (-a') = 0 \\ a + (-a') &> a' + (-a') = 0 \\ -a + a'' &> -a'' + a'' = 0. \\ \text{Thus} \quad & a + (-a) = 0. \end{aligned}$$

We can verify that addition of surreal numbers is associative and thus satisfy the axioms for an abelian group under addition. Let $a = A'|A''$, $b = B'|B''$, and $c = C'|C''$.

$$\begin{aligned} & (a+b)+c = \{a+b', b+a'\}|\{a+b'', b+a''\} + c \\ = \quad & \{(a+b)+c', c+(a+b'), c+(b+a'\}|\{(a+b)+c'', c+(a+b''), c+(b+a'')\} \\ = \quad & \{a+(b+c'), a+(c+b'), (b+c)+a'\}|\{a+(b+c''), a+(c+b''), (b+c)+a''\} \\ = \quad & a + \{b+c', c+b'\}|\{b+c'', c+b''\} \\ = \quad & a + (b+c). \end{aligned}$$

The Multiplication of Surreal Numbers

We now define multiplication.

As a motivation for the following definition consider real numbers a, b, c, d where $a < b$, and $c < d$. We then have $b - a > 0$ and $d - c > 0$. Thus

$(b-a)(d-c) > 0 \Rightarrow bd+ac-ad-bc > 0 \Rightarrow bd > ad+bc-ac$, and $bc < bd+ac-ad$.

Definition Let $a = A'|A''$, $b = B'|B''$, then

$$ab = \{a'b + ab' - a'b', a''b + ab'' - a''b''\}|\{a'b + ab'' - a'b'', a''b + ab' - a''b'\}$$

where $a' \in A', b' \in B', a'' \in A''$ and $b'' \in B''$.

Exercise Consider $\epsilon = (+ - - - \cdots) = \{0\}|\{1, \frac{1}{2}, \frac{1}{4}, \cdots\}$ and
$\omega = (+ \cdots) = \{0, 1, \cdots\}|\emptyset$. Find $\epsilon \cdot \omega$.

The Field Properties of Surreal Numbers

We have confirmed that Surreal numbers with the binary operation of addition forms an abelian group. We now want to complete our verification that Surreal numbers with the operations of addition and multiplication form a field. Of course we must verify that the definition of multiplication does indeed define a Surreal number, to do that we must first verify the properties of commutativity and associativity of multiplication, and the distributive property.

We again make the inductive hypothesis that the inequalities necessary are valid for the product of surreal numbers whose sum of lengths are less than the sum of the lengths of a and b, and thus the definition of multiplication will be well defined.

Let $a = A'|A''$, $b = B'|B''$ and $c = C'|C''$. We now have

$$\begin{aligned} ab = & \{a'b + ab' - a'b', a''b + ab'' - a''b''\}|\{a'b + ab'' - a'b'', a''b + ab' - a''b'\} \\ = & \{b'a + ba' - b'a', b''a + ba'' - b''a''\}|\{b'a + ba'' - b'a'', b''a + ba' - b''a'\} = ba \end{aligned}$$

Where $a' \in A', a'' \in A'', b' \in B'$, and $b'' \in B''$. Thus multiplication is commutative.

$$
\begin{aligned}
(ab)c = \quad & \{(ab)'c + (ab)c' - (ab)'c', (ab)''c + (ab)c'' - (ab)''c''\}| \\
& \{(ab)'c + (ab)c'' - (ab)'c'', (ab)''c + (ab)c' - (ab)''c'\} \\
= \quad & \{(a'b + ab' - a'b')c + abc' - (a'b + ab' - a'b')c', \\
& (a''b + ab'' - a'b'')c + abc' - (a''b + ab'' - a''b'')c', \\
& (a'b + ab'' - a'b'')c + abc'' - (a'b + ab'' - a'b'')c'', \\
& (a''b + ab' - a''b')c + abc'' - (a''b + ab' - a''b')c''\}| \\
& \{(a'b + ab' - a'b')c + (ab)c'' - (a'b + ab' - a'b')c'', \\
& (a''b + ab'' - a''b'')c + (ab)c'' - (a''b + ab'' - a''b'')c'', \\
& (a'b + ab'' - a'b'')c + (ab)c' - (a'b + ab'' - a'b'')c', \\
& (a''b + ab' - a''b')c + (ab)c' - (a''b + ab' - a''b')c'\} \\
= \quad & \{a'bc + ab'c - a'b'c + abc' + abc' - a'bc' - ab'c' + a'b'c', \\
& a''bc + ab''c - a''b''c + abc' - a''bc - ab''c + a''b''c, \\
& a'bc + ab''c - a'b''c + abc'' - a'bc'' - ab''c + a'b''c'', \\
& a''bc + ab'c - a''b'c + abc'' - a''bc'' - ab'c'' + a''b'c''\}| \\
& \{a'bc + ab'c - a'b'c + abc'' - a'bc'' - ab'c'' + a'b'c'', \\
& a''bc + ab''c - a''b''c + abc'' - a''bc'' - ab''c'' + a''b''c'', \\
& a'bc + ab''c - a'b''c + abc' - a'bc' - ab''c' + a'b''c', \\
& a''bc + ab'c - a''b'c + abc' - a''bc' - ab'c + a''b'c' \\
= \quad & \{a'(bc) + a(b'c + bc' - b'c') - a'(b'c + bc' - b'c'), \\
& a'(bc) + a(b''c + bc'' - b''c'') - a'(b''c + bc'' - b''c''), \\
& a''(bc) + a(b'c + bc' - b'c') - a''(b'c + bc - b'c'), \\
& a''(bc) + a(b''c + bc'' - b''c'') - a''(b''c + bc'' - b''c'')\}| \\
& \{a'(bc) + a(b'c + bc'' - b'c'') - a'(b'c + bc'' - b'c''), \\
& a'(bc) + a(b''c + bc' - b''c') - a'(b''c + bc' - b''c'), \\
& a''(bc) + a(b'c + bc' - b'c') - a''(b'c + bc' - b'c'), \\
& a''(bc) + a(b''c + bc'' - b''c'') - a''(b''c + bc'' - b''c'')\}
\end{aligned}
$$

$$
\begin{aligned}
&= \{a'(bc)+a(bc)'-a'(bc)',a''(bc)+a(bc)''-a''(bc)''\}| \\
&\quad \{a'(bc)+a(bc)''-a'(bc)'',a''(bc)'-a''(bc)'\}=a(bc)
\end{aligned}
$$

Where $a' \in A', a'' \in A'', b' \in B', b'' \in B'', c' \in C'$, and $c'' \in C''$. Thus multiplication is associative.

$$
\begin{aligned}
a(b+c) = &\ \{a'(b+c)+a(b+c)'-a'(b+c)',a''(b+c)+a(b+c)''-a''(b+c)''\}| \\
&\ \{a'(b+c)+a(b+c)''-a'(b+c)'',a''(b+c)+a(b+c)'-a''(b+c)'\} \\
= &\ \{a'(b+c)+a(b+c')-a'(b+c'),a'(b+c)+a(b'+c)-a'(b'+c), \\
&\ a''(b+c)+a(b+c'')-a''(b+c''),a''(b+c)+a(b''+c)-a''(b''+c)\}| \\
&\ \{a'(b+c)+a(b+c'')-a'(b+c''),a'(b+c)+a(b''+c)-a'(b''+c), \\
&\ a''(b+c)+a(b+c')-a''(b+c'),a''(b+c)+a(b'+c)-a''(b'+c)\} \\
= &\ \{a'b+a'c+ab+ac'-a'b'-a'c',a'b+a'c+ab'+ac-a'b'-a'c, \\
&\ a''b+a''c+ab+ac''-a''b-a''c'',a''b+a''c+ab''+ac-a''b''-a''c\}| \\
&\ \{a'b+a'c+ab+ac''-a'b-a'c'',a'b+a'c+ab''+ac-a'b''-a'c, \\
&\ a''b+a''c+ab+ac'-a''b-a''c',a''b+a''c+ab'+ac-a''b'-a''c\} \\
= &\ \{ab+a'c+ac'-a'c',ac+a'b+ab'-a'b', \\
&\ ab+a''c+ac''-a''c'',ac+a''b+ab''-a''b''\}| \\
&\ \{ab+a'c+ac''-a'c'',ac+a'b+ab''-a'b'', \\
&\ ab+a''c+ac'-a''c',ac+a''b+ab'-a''b'\} \\
= &\ \{ab+(ac)',(ab)'+ac\}|\{ab+(ac)''+(ab)''+ac\}=ab+ac
\end{aligned}
$$

Where $a' \in A', a'' \in A'', b' \in B', b'' \in B'', c' \in C'$, and $c'' \in C''$. Thus multiplication distributes over addition.

We now verify that the Surreal number $1 = \{0\}|\{\}$ is the multiplicative identity. First we must verify that any surreal number multiplied times 0 is 0. Let $a = A'|A''$, and we have $0 = \{\}|\{\}$, then we have

$$a \cdot 0 = \{a'b + ab' - a'b', a''b + ab'' - a''b''\}|\{a'b + ab'' - a'b'', a''b + ab' - a''b'\}$$

where $a' \in A', b' \in \{\}, a'' \in A''$ and $b'' \in \{\}$

Since there are no elements in $\{\}$ we conclude that the left and right sets of the product are empty and thus is the Surreal number 0.

Now consider $a \cdot 1$.

$$a \cdot 1 = \{a' \cdot 1 + a \cdot 0 - 1 \cdot 0\}|\{a'' \cdot 1 + a \cdot 0 - a'' \cdot 0\} = \{a'\}|\{a''\} = a.$$

Hence 1 is the multiplicative identity.

To demonstrate that all non-zero surreal numbers have multiplicative inverses is considerably more technical than the calculation arguments used for the associative and distributive properties. A complete development of inverses is given in the book "An introduction to the theory of Surreal Numbers" by Harry Gonshor. We give here an intuitive argument as to why inverses should exist.

Given a non-zero surreal number x we naively pick a candidate $y_0 = B'|B''$ for its inverse. $y_0 > 0$ if $x > 0$ and $y_0 < 0$ if $x < 0$. If $x \cdot y_0 > 1$, then we construct our next candidate by $y_1 = B'|B'' \cup \{y_0\}$ if $x > 0$ or $y_1 = B' \cup \{y_0\}|B''$ if $x < 0$. If $x \cdot y_0 < 1$, then we construct our next candidate by $y_1 = B' \cup \{y_0\}|B''$ if $x > 0$ or $y_1 = B'|B'' \cup \{y_0\}$ if $x < 0$.

We proceed in this fashion until the product is in fact 1. We naively believe that the procedure will eventually end as we will exhaust all possible numbers that could lie between the sets of $x \cdot y_\alpha$ except for 1. Then y_α will be the multiplicative inverse of x.