X TRANSFINITE AND INFINITESIMAL NUMBERS

In his development of the Calculus, Isaac Newton introduced the concept of a “fluxion”, a number that was smaller than any positive number and yet was greater than zero. This concept seemed to be contrary to the Archimedian principle, and led one of the critics of early analysis, Bishop Berkeley, to comment “And what are these fluxions? The velocities of evanescent increments. And what are these same evanescent increments? They are neither finite quantities, nor quantities infinitely small, nor yet nothing. May we not call them ghosts of departed quantities?” At this time in history, when Newton’s genius is greatly admired we may be inclined to dismiss Bishop Berkeley’s comments as those of a pompous fool. However this is not true, his criticisms were well founded as the rigorous structure of Newton’s Calculus had not been well established. Criticism such as Bishop Berkeley, is good for the development of any scholarly field of study, as it forces those involved in that study to construct a firm basis for their claims. Of course the concept of fluxions has been replaced with the limit, and all seems well with the world, and Calculus.

With Cantor’s development of transfinite ordinal and cardinal numbers it has become possible to make the concept of fluxions rigorous, and without contradicting Archimedes. We will not use the term fluxion, but will call such numbers infinitesimal. In this chapter we will construct transfinite Integers, Rational and Real Numbers. The construction will be a simple extension of our development of finite numbers.

We begin our construction of transfinite and infinitesmals by noting that our construction of the integers, rational and real numbers used the ordinal number ω as a basis on which to build the appropriate collection of sets that would be our numbers. An attempt to use a larger ordinal than ω will fail, as ordinal arithmetic is not commutative, and commutativity is crucial in the construction of the equivalence classes that represent the different numbers. Consider the equivalence relation defined on $\omega \times \omega$ that yielded the integers.

$$(a, b) \equiv (c, d) \Leftrightarrow a + d = b + c.$$

This relation is not an equivalence relation on ordinal numbers greater than ω as $(\omega, 1) \not\equiv (\omega, 1)$ since $\omega + 1 \neq 1 + \omega$, hence the relation does not satisfy the reflexive property.

Transfinite Arithmetic

Our choices of binary operations were made from our experiences with tangible collection of objects (e.g. boxes of apples), and these tangible collections must be finite by their very nature. The extension of these binary operations to abstract and infinite sets, although well motivated is quite arbitrary. So our strategy is to define a new arithmetic, that agrees with ordinal arithmetic on finite sets, but satisfies the appropriate field axioms on infinite sets. As noted in Chapter 3, the ordinal number ω^ω can be expressed as

$$\{x \mid x = \sum_{i=0}^{n} \omega^i \alpha_i \text{ where } n, \alpha_i \in \omega\}.$$

Thus every element of ω^ω is of the form of a polynomial of indeterminate ω with coefficients and exponents in ω. We thus define an arithmetic on ω^ω to be polynomial arithmetic.

$$\sum_{i=0}^{n} \omega^i \alpha_i + \sum_{i=0}^{m} \omega^i \beta_i = \sum_{i=0}^{max\{n,m\}} \omega^i (\alpha_i + \beta_i)$$

and

$$\sum_{i=0}^{n} \omega^i \alpha_i \ \cdot \ \sum_{j=0}^{m} \omega^j \beta_j = \sum_{i=0}^{n} \sum_{j=0}^{m} \omega^{i+j} (\alpha_i \cdot \beta_j).$$

Where the arithmetic on ω is ordinal arithmetic.

We may now easily verify that these two operations are commutative and associative, and that multiplication distributes over addition. Thus we may repeat our construction of integers, rational and real numbers with this arithmetic to form transfinite integers, transfinite rational, and transfinite real numbers. Both the transfinite rational numbers and transfinite real numbers contain elements that satisfy the conditions of Newton's fluxions. We will call those numbers infinitesimal.

If we let $n \in \omega$ be an arbitrary element , and we let ω represent the transfinite integer $(\omega, 0)$, 1 represent the transfinite integer $(1, 0)$ and n represent the transfinite integer $(n, 0)$ we observe that $1 \cdot n < 1 \cdot \omega \Rightarrow [1, \omega] < [1, n]$, where $[1, \omega]$ and $[1, n]$ are transfinite rational numbers. Since the restriction of the construction of transfinite rational numbers is the construction of the rational numbers, we may say that $[1, n]$ is a rational number. Thus for any positive real number ϵ, by the Archimedian property there exists an integer n such that $[1, n] < \epsilon$, and since $[1, \omega] < [1, n]$ we have $[1, \omega] < \epsilon$. And thus $[1, \omega]$ satisfies the condition of Newton's fluxions.

Transfinite Numbers

We define the ω-**Transfinite Natural Numbers**, $\mathbb{N}_\omega$ to be ω^ω, with polynomial arithmetic. We define the ω-**Transfinite Integers**, $\mathbb{Z}_\omega$, to be the collection of equivalence classes

$$\{[x, y] | x, y \in \mathbb{N}_\omega \text{ where } (x, y) \equiv (z, w) \ \Leftrightarrow \ x + w = y + z\}.$$

We define the **ω-Transfinite Rational numbers**, $\mathbb{Q}_\omega$, to be the collection of equivalence classes

$$\{[x,y]|x,y \in \mathbb{Z}_\omega \text{ where } (x,y) \equiv (z,w) \;\Leftrightarrow\; xw = yz\}.$$

And we define the **ω-Transfinite Real Numbers**, $\mathbb{R}$, to be the collection of all Dedekind cuts of transfinite rational numbers.

We now observe that the ordinal number $\omega^{(\omega^\omega)}$ can be expressed as

$$\{x \mid x = \sum_{i=0}^{n} (\omega^\omega)^i \alpha_i \text{ where } n, \alpha_i \in \omega^\omega\}.$$

Thus every element of ω^ω is of the form of a polynomial of indeterminate ω^ω with coefficients and exponents in ω^ω. We again define arithmetic as polynomial arithmetic where the arithmetic on ω^ω is the arithmetic previously defined.

We now define the ω^ω-Transfinite Natural Numbers, ω^ω-Transfinite Integers, ω^ω-Transfinite Rational Numbers, and ω^ω-Transfinite Real Numbers in the analogous fashion. We may continue indefinitely constructing Transfinite numbers in this fashion, We thus define any number constructed in this fashion to be a Transfinite Number. Thus a Transfinite Integer is an α-Transfinite Integer for some appropriate ordinal number α. And equivalently for Transfinite Natural Numbers, Transfinite Rational Numbers, and Transfinite Real numbers.

We will adopt the convention, that when referring to an element of the transfinite real numbers that is the embedded image of either a transfinite natural number, transfinite integer, or transfinite rational number, we will simply refer to that number as being an element of that respective embedded set, that is, a transfinite natural number, transfinite integer or transfinite rational number.

Let ξ be an arbitrary transfinite real number. Consider the cut $\{\alpha|\alpha^2 < \xi\}$. We may consider this cut to represent the transfinite real number $\sqrt{\xi}$.

Questions What is $\sqrt{\omega}$? What is $\sqrt{\omega+1}$? What is $\sqrt[3]{\omega}$?