VI ARITHMETIC

In this chapter we will develop the concept of arithmetic for ordinal and cardinal numbers. It is conceptually easier to define an arithmetic for cardinal numbers, so we will do that first and extend those concepts to ordinal numbers.

Cardinal Arithmetic

Definition A **Binary Operation** on a set a is a function from $a \times a$ to a.

Definition An **Arithmetic** on a set a is a collection of binary operations on the set a. The collection is usually finite.

We extend these two definitions to include arbitrary collections that are not necessarily sets.

Definition A **Binary Operation** on any collection is a rule that assigns to every ordered pair of elements of the collection a unique element of the collection.

The definition for an arithmetic on an arbitrary collection is identical except the term set is replaced with collection.

We will represent every binary operation by a symbol, and indicate the element to which any ordered pair of elements is associated by the two elements juxtaposed with the operation symbol inserted between them. For Example, if $*$ represents the binary operation and a and b are elements of the collection then $a * b$ will represent the element to which the ordered pair (a, b) is associated.

We now define an arithmetic for the collection of Cardinal Numbers. The first operation we will define we will call **addition**, which will be represented by the symbol $+$, and the associated element to the ordered pair will be known as the **sum**. The motivation for the definition is naive and intuitive. We would like to say that the sum of the cardinality of two sets is the cardinality of the union of the two sets, but this is a little too naive. The two sets may have non-empty intersection, and we wish to avoid this situation.

Let a and b be two arbitrary sets and $\{0, 1\}$ be a two point space. The set $a \times \{0\}$ is cardinally equivalent to a by the obvious bijection, as is $b \times \{1\}$ with b. $a \times \{0\}$ and $b \times \{1\}$ are disjoint sets, as the second member of each element of one set is different from the second member of each element of the other set.

Definition The **Disjoint Union** of two sets a and b, which will be denoted $a \uplus b$, is $a \times \{0\} \ \cup \ b \times \{1\}$.

We may now properly define cardinal addition.

Definition Let A and B be cardinal numbers. There are two sets a and b such that $C(a) = A$ and $C(b) = B$, and we define $A + B$ to be $C(a \uplus b)$.

The second binary operation we define on cardinal numbers we will call **multiplication**. We will nominally use the symbol $\cdot$ for multiplication, and the associated element to the ordered pair will be known as the **product**. When indicating the product it is unambiguous to omit the symbol for multiplication and simply indicate the product by the juxtaposition of the two elements of the ordered pair, e.g., AB will represent $A \cdot B$.

The motivation for the definition for cardinal multiplication is nearly as intuitive as that for addition. We imagine that we have a collection of sets of equal cardinality and we wish to determine the cardinality of the total collection. Every element of the total collection can be represented by an ordered pair, the first member is the symbol for that element within its set, and the second member is the symbol for the set to which it belongs.

Definition Let A and B be cardinal numbers. There are sets a and b such that $C(a) = A$ and $C(b) = B$, and we define $AB = C(A \times B)$.

The third binary operation that we shall define on cardinal numbers we shall call **exponentiation**. We will nominally use the symbol $\wedge$ for exponentiation and the associated element to the ordered pair will be known as the **power**. Again we choose to use a different but still unambiguous notation for common use, we will use the second member as a superscript to indicate the power. I.e., $a^b = a \wedge b$.

The motivation for the definition of cardinal exponentiation is that we imagine that we have an arbitrary collection of arbitrary collections of sets of equal cardinality, and we wish to determine its cardinality. Recall that the cartesian product of a collection of sets is the collection of all maps from the collection of sets to the union of all elements of the sets, where the image of a set is restricted to the elements of itself. Thus our model for exponentiation is a collection of duplications of a given set, and we wish to compute the cardinality of the cartesian product of this collection.

Definition Let A and B be cardinal numbers. There are sets a and b such that $C(a) = A$ and $C(b) = B$, and we define $A^B = C(\{f | f : b \to a\})$.

Ordinal Arithmetic

We will now define an arithmetic for ordinal numbers. We wish to extend the concept of addition relating to the union of two sets, but we also wish to preserve the order properties of ordinal numbers.

Recall that in Chapter V we defined **order type** to be the unique ordinal number to which a well ordered set is order isomorphic, and we use the abbreviation OT to refer to the order type.

The first operation we will define will again be called addition and its associated symbol will also be $+$.

Let a and b be ordinal numbers, and $\{0,1\}$ be a two point space. The sets $a \times \{0\}$ and $b \times \{1\}$ are similar to a and b by the obvious bijection.

Definition If a and b are ordinal numbers, then $a + b = OT(a \uplus b)$, where $(x,0) < (y,0)$ iff $x < y$, $(x,1) < (y,1)$ iff $x < y$, and $(x,0) < (y,1)$ $\forall\ x \in a$ and $\forall\ y \in b$.

The second operation defined will again be called multiplication and is associated symbol will also be $\cdot$.

We want to extend the concept of cardinal multiplication so we will concern ourselves with the cartesian product of ordinal numbers. We wish to extend the order of the ordinal numbers in a fashion that will well order the cartesian product. Let a and b be ordinal numbers. We define an order relation on $a \times b$ by

$$(x,y) < (z,w) \text{ iff } \begin{cases} y < w \quad \text{or} \\ \text{if } y = w \text{ then } x < z. \end{cases}$$

This order is called reverse lexicographic order.

Definition If a and b are ordinal numbers, then $a \cdot b = OT(a \times b)$, where $(a \times b)$ is well ordered by reverse lexicographic order.

Verifying that reverse lexicographic order is a well ordering is straightforward. Pick any non-empty subset of the cartesian product $a \times b$. The collection of second members of the elements of this subset has a least element. Pick all those elements that has that least second member, from those pick the element that has the least first member. This will be the least element of the chosen subset. Hence reverse lexicographic ordering is a well ordering of the cartesian product of two ordinal numbers.

Lemma For ordinal numbers a, b, c, if $a = b$, then $a + c = b + c$ and $ac = bc$.

Before we prove the lemma we give the following definition and observation.

Definition The function from a set a to itself defined by $I : x \to x \; \forall \; x \in a$ is called the **Identity map**.

We observe that if $a = b$, then $a \uplus c = b \uplus c$. We can now prove the lemma.

Proof The identity maps $I_1 : a \uplus c \to b \uplus c$ and $I_2 : a \times c \to b \times c$ are order preserving bijections. Let $\beta_1 : OT(a \uplus c) \to a \uplus c$, $\gamma_1 : OT(b \uplus c) \to b \uplus c$, $\beta_2 : OT(a \times c) \to a \times c$, $\gamma_2 : OT(b \times c) \to b \times c$ be order preserving bijections. We now have that $\gamma_1^{-1} \circ I_1 \circ \beta_1 : OT(a \uplus c) \to OT(b \uplus c)$ and $\gamma_2^{-1} \circ I_2 \circ \beta_2 : OT(a \times c) \to OT(b \times c)$ are order preserving bijections. ∎

We now define the third operation which we call exponentiation and we also choose the associated symbol $\wedge$, and again we abbreviate with superscripting.

Exponentiation is defined recursively by

i. $a \wedge 0 \equiv a^0 = 1$,

ii. $a \wedge (b+1) \equiv a^{b+1} = a^b \cdot a$,

iii. $a \wedge b \equiv a^b = \sup\{a^c | c < b\}$ if b is a limit ordinal.

Exercises Let a, b, c be ordinal numbers, show

1. $a^b \cdot a^c = a^{b+c}$.

2. $(a^b)^c = a^{b \cdot c}$.

Hint: Let d be any ordinal number containing c, e.g. $c+1$. For 1 Let $A = \{y \in d | a^b \cdot a^y = a^{b+y}\}$, and use transfinite induction to show $A = d$. For 2 let $A = \{y \in d | (a^b)^y = a^{b \cdot y}\}$.

Since Cardinal numbers and Ordinal numbers are the same in ω the reader should verify that ordinal and cardinal arithmetic agree.

We also leave it to the reader to verify these arithmetic facts.

1. $\aleph_0 + \aleph_0 = \aleph_0$.

2. $\aleph_0 \cdot \aleph_0 = \aleph_0$.

3. $\aleph_0 \wedge \aleph_0 > \aleph_0$.

4. $n + \omega = \omega \ \forall n \in \omega$.

5. $\omega + n = \omega + n \ \forall n \in \omega$, where the left side of the equation refers to ordinal addition while the right side refers to the ordinal number $\omega + n$.

6. $n \cdot \omega = \omega \ \forall n \in \omega$.

7. $\omega \cdot n = \omega n \ \forall n \in \omega$.

We invite the reader to either confirm or deny any arithmetic fact that he may hypothesize; i.e. add or multiply a few numbers and see what happens.

Definition Let a and b be elements of ω. If $b = c+1$, then we define $b-1 \equiv c$.

Lemma If $a, b \in \omega$ and $C(a) = C(b)$, then $a = b$.

Proof Let $c = \{y \in \omega | C(y) < C(y+1)\}$, and let $x \in \omega$ such that $\mathrm{S}(x) \subset c$. If $x = 0$, we then have $C(0) < C(1)$, thus $0 \in c$. If $x \neq 0$, we then have $x - 1 \in c$, and thus $C(x-1) < C(x)$.

We can clearly establish that there is no bijection between 0 and 1, or 1 and 2. Now for $x > 1$ assume there exists a bijection $\beta : x \to x+1$. We may then create a bijection $\gamma : x-1 \to x$ by

$$\gamma(y) = \begin{cases} \beta(y) & \text{if } y \neq \beta^{-1}(x) \\ \beta(x-1) & \text{if } y = \beta^{-1}(x). \end{cases}$$

Thus $C(x-1) = C(x)$, which is a contradiction. Thus $C(x) < C(x+1)$, and thus $x \in c$, and thus by transfinite induction $C(x) < C(x+1)\ \forall\ x \in \omega$. By transitivity we have $a < b \ \Rightarrow\ C(a) < C(b)$.

Thus $C(a) = C(b) \ \Rightarrow\ a = b$. ∎

An alternate definition for addition and multiplication on ω is

$$\begin{aligned} a + 0 &= a \\ a + b &= (a+1) + (b-1) \end{aligned}$$

and

$$\begin{aligned} a \cdot 0 &= 0 \\ a \cdot b &= a + a \cdot (b-1). \end{aligned}$$

Corollary This definition for addition and multiplication agrees with ordinal arithmetic on ω.

Proof Using the previous lemma we need only demonstrate equivalent cardinality. Define a bijection $\beta : a \uplus b \to (a+1) \uplus (b-1)$ by

$$\beta(x) = \begin{cases} x & \text{if } x \in a \times \{0\} \\ x & \text{if } x \in (b-1) \times \{1\} \\ (a,0) & \text{if } x = ((b-1),1). \end{cases}$$

We also note that $C(a+0) = C(a \cup \emptyset) = C(a)$.

To show $a \cdot b = a + a \cdot (b-1)$ we define the bijection $\beta(a \times b) \to a \uplus a \times (b-1)$ by

$$\beta(x,y) = \begin{cases} (x,0) & \text{if } y = b-1 \\ ((x,y),1) & \text{if } y \in b-1. \end{cases}$$

Also $a \cdot 0 = C(a \times \emptyset) = C(\emptyset) = 0$.

Exercises

For $a, b, c \in \omega$ show the following:

1. $(a+b)+c = a+(b+c)$.

2. $a+b = b+a$.

3. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

4. $a \cdot b = b \cdot a$.

5. $a \cdot 1 = a$.

6. $(a+b)\cdot c = a\cdot c + b\cdot c$.

Solution to 1 define the bijection

$$\beta : (a\times\{0\}\uplus b\times\{1\})\times\{0\}\uplus c\times\{1\} \to a\times\{0\}\uplus(b\times\{0\}\uplus c\times\{1\})\times\{1\}$$

by

$$\beta((x,n,0)) = \begin{cases} (x,0) & \text{if } x\in a \\ (x,0,1) & \text{if } x\in b \end{cases}$$

and $$\beta((x,1)) = (x,1,1) \text{ if } x\in c.$$