V ZORN'S LEMMA AND WELL ORDERING

We have shown that there are sets with cardinality greater than $\aleph_0$ but we have yet to demonstrate that there exists ordinal numbers with cardinality greater than $\aleph_0$. We will accomplish this task by showing that every set can be well ordered, and that every well ordered set is order isomorphic to an ordinal number. By order isomorphic we shall mean the following.

Definition Two partially ordered sets a and b are said to be **order isomorphic** if there exists a bijection between them that preserves order. That is if $\beta : a \longrightarrow b$ is the order preserving bijection then $x \leq y$ if and only if $\beta(x) \leq \beta(y)$. We write $a \simeq b$.

Zorn's Lemma

Theorem 5.1 *Zorn's Lemma*: If X is a partially ordered set such that every chain in x has an upper bound in X, then X contains a maximal element.

By chain we mean a totally or linearly ordered subset. In the hypothesis of Zorn's Lemma the upper bound need not be in the chain. In the conclusion the maximal element is simply an element with no superior, that is if $x \leq y \Rightarrow x = y$ then x is maximal. There may in fact be elements in X that are not comparable to x, yet x may still be maximal.

Proof Let X be a partially ordered set. For each $x \in X$, $\bar{\mathrm{S}}(x) = \{y \in X \mid y \leq x\}$ is the weak section of x. $\bar{\mathrm{S}}$ is a function from X to $\mathcal{P}(X)$ since the section of any element is unique. The range R of $\bar{\mathrm{S}}$ is a collection of subsets that are partially ordered by inclusion, i.e. $A \leq B$ if $A \subseteq B$. $\bar{\mathrm{S}}$ is one to one, since $\bar{\mathrm{S}}(x) \subseteq \bar{\mathrm{S}}(y)$ if and only if $x \leq y$. Thus if we find a maximal element $\bar{\mathrm{S}}(z)$ in R then z is maximal in X. Also C is a chain in X if and only if $\bar{\mathrm{S}}(C)$ is a chain in R.

Let $\mathcal{X}$ be the set of all chains in X. Let $\Gamma \in \mathcal{X}$. Since Γ is a chain it has an upper bound x by hypothesis thus $\Gamma \subseteq \bar{\mathrm{S}}(x)$ for some $x \in X$. $\mathcal{X}$ is a non-empty collection of sets partially ordered by inclusion. Now if C is a chain in $\mathcal{X}$ then $G = \bigcup_{\Gamma \in C} \Gamma \in \mathcal{X}$. G is an upper bound of C in $\mathcal{X}$ as each element of C is dominated by G.

Now let $\mathbf{X}$ be an arbitrary non-empty collection of subsets of a non-empty set X subject to

1. if $A \in \mathbf{X}$ and $B \subseteq A$ then $B \in \mathbf{X}$, and
2. if C is a chain in $\mathbf{X}$ then $\bigcup_{\Gamma \in C} \Gamma \in \mathbf{X}$.

Notice that these are the exact conditions that our set $\mathcal{X}$ had in the previous discussion. Also notice that the first condition implies that $\emptyset \in \mathbf{X}$. Our task is to show that $\mathbf{X}$ has a maximal element.

Now let ϕ be a choice function from the non-empty subsets of X to X, i.e $\phi : (\mathcal{P}(X) - \emptyset) \longrightarrow X$. We note that ϕ is a function such that $\phi(A) \in A$ for all non-empty subsets A of X. For each set $A \in \mathbf{X}$ let $\hat{A} = \{x \in X \mid A \cup \{x\} \in \mathbf{X}\}$. Define a function $\gamma : \mathbf{X} \longrightarrow \mathbf{X}$ by the following:

$$\gamma(A) = \begin{cases} A \cup \{\phi(\hat{A} - A)\} & \text{if } \hat{A} - A \neq \emptyset \\ A & \text{if } \hat{A} - A = \emptyset. \end{cases}$$

We observe that $\hat{A} - A = \emptyset$ if and only if A is maximal. Our task is now to show there exists a set A in $\mathbf{X}$ such that $\gamma(A) = A$. Since $\phi(\hat{A} - A)$ is a single element we notice that $\gamma(A)$ contains at most one more element than A.

We now define a **tower** as a subcollection T of $\mathbf{X}$ that satisfies the following conditions:

1. $\emptyset \in T$,
2. if $A \in T$, then $\gamma(A) \in T$, and
3. if C is a chain in T, then $\bigcup\limits_{A \in C} A \in T$.

We notice here that $\mathbf{X}$ satisfies the conditions for a tower, and thus towers exist. We can also easily verify that the intersection of a collection of towers is a tower. Let $\{T_\lambda\}$ be a collection of towers, since $\emptyset \in T_\lambda$ for all λ, $\emptyset \in \bigcap\limits_{\lambda} T_\lambda$. If $A \in \bigcap\limits_{\lambda} T_\lambda$, then $A \in T_\lambda$ for all λ, thus $\gamma(A) \in T_\lambda$ for all λ and thus $\gamma(A) \in \bigcap\limits_{\lambda} T_\lambda$ for all λ. And finally if C is a chain in $\bigcap\limits_{\lambda} T_\lambda$, then C is a chain in T_λ for all λ, thus $\bigcup\limits_{A \in C} A \in T_\lambda$ for all λ, and thus $\bigcup\limits_{A \in C} A \in \bigcap\limits_{\lambda} T_\lambda$. It follows that the intersection of all towers T_0 is the smallest tower. We now wish to show that T_0 is a chain.

We say that a set B in T_0 is **comparable** if it is comparable with every set in T_0, that is, for all $A \in T_0$ either $A \subseteq B$ or $B \subseteq A$. To show that T_0 is a chain we show that every set in T_0 is comparable.

We now let B be an arbitrary comparable set in T_0. Comparable sets do exist since $\emptyset$ is clearly comparable. Suppose $A \in T_0$ and A is a proper subset of B. Since B is comparable we have $\gamma(A) \subseteq B$ or B is a proper subset of $\gamma(A)$. If $B \subset \gamma(A)$ we have A as a proper subset of B and B a proper subset of $\gamma(A)$, but $\gamma(A) - A$ is a singleton thus there cannot be a set between them. We conclude $\gamma(A) \subseteq B$.

Now consider the collection U of all sets A in T_0 where either $A \subseteq B$ or $\gamma(B) \subseteq A$. The collection U is no larger than the collection of sets in T_0 that are comparable with $\gamma(B)$ since, if $A \in U$ and since $B \subseteq \gamma(B)$ we have either $A \subseteq \gamma(B)$ or $\gamma(B) \subseteq A$.

We now claim that U is a tower. We verify the three conditions:

1) $\emptyset \in U$.

2) To show $A \in U \Rightarrow \gamma(A) \in U$ Consider three cases

i. $A \subset B$.

ii. $A = B$.

iii. $\gamma(B) \subseteq A$.

For i. $\gamma(A) \subseteq B$ by the preceding argument thus $\gamma(A) \in U$.

For ii. $\gamma(A) = \gamma(B)$ thus $\gamma(B) \subseteq \gamma(A)$, therefore $\gamma(A) \in U$.

For iii. $\gamma(B) \subseteq A \Rightarrow \gamma(B) \subseteq \gamma(A)$, therefore $\gamma(A) \in U$.

3) Let C be a chain in U, if $\gamma(B) \subseteq D$ for some $D \in C$, then $\gamma(B) \subseteq \bigcup_{D \in C} D$ hence $\bigcup_{D \in C} D \in U$. If however $D \subseteq B$ for all $D \in C$, then $\bigcup_{D \in C} D \subseteq B$. Thus we may conclude $\bigcup_{D \in C} D \in U$.

We thus conclude that U is a tower and is a subset of T_0 which is the smallest tower hence we have $U = T_0$.

Now let B be a comparable set, we form U as above and since $U = T_0$ for any $A \in T_0$ we have either $A \subseteq B \subseteq \gamma(B)$ or $\gamma(B) \subseteq A$. We thus conclude that if B is a comparable set, then $\gamma(B)$ is comparable also.

We have that $\emptyset$ is comparable and γ maps comparable sets to comparable sets. Now since the union of a chain of comparable sets is comparable we may conclude that the comparable sets constitutes a tower, and hence they exhaust all of T_0.

T_0 is a chain, thus if $A = \bigcup_{B \in T_0} B \in T_0$ we have $\gamma(A) \subseteq A$, since the union A includes all the sets in T_0. We always have $A \subseteq \gamma(A)$, thus we conclude that $A = \gamma(A)$. This is the condition that we noted earlier needed to be shown to complete the proof. ■

Definition A well ordered set A is a **continuation** of a well ordered set B if

i) $B \subset A$

ii) $B = \mathrm{S}(a)$ for some $a \in A$

iii) For $a, b \in B$, $a \leq_B b$ iff $a \leq_A b$.

The reason for the third condition is that a set may have more than one ordering, and for continuation we want B to have the same ordering as A when restricted to B.

The Well Ordering and Counting Theorems

Theorem 5.2 *The Well Ordering Theorem* Every set can be well ordered.

An important point to be noted here is that the set may be presented with an ordering that is not a well ordering. The Well Ordering Theorem says we may disregard any previously assigned ordering that the set may have and endow it with a new ordering that is a well ordering.

Before we prove this theorem we make this note. We shall regard an ordered set X as the pair $(X, <)$ where $<$ is an order relation.

Proof Let X be a set. Let W be the collection of all well ordered subsets of X under every possible ordering, i.e.

$$W = \{(A, <) \in \mathcal{P}(X) \times \mathcal{P}(A \times A) \mid < \text{ is a well ordering of } A\}.$$

We partially order W by continuation, i.e. $(A, <) <_W (B, <)$ if B is a continuation of A. W is not empty since $(X, <)$ where $<= \{(x, x) \mid x \in X\}$ is an element of W.

Now let C be a chain in W,

$$C = \{(A_\lambda, <_\lambda) \mid (A_{\lambda_i}, <_{\lambda_i}) \ <_W (A_{\lambda_k}, <_{\lambda_k}) \text{ for } \lambda_i < \lambda_k\}.$$

Since the A_λ are nested sets, $\bigcup\limits_{A_\lambda \in C} A_\lambda$ is an upper bound for C, and is in W since any subset of $\bigcup\limits_{A_\lambda \in C} A_\lambda$ must be a subset of A_{λ_i} for some λ_i, and thus have a least element, and therefore be well ordered.

Hence the condition for the hypothesis of Zorn's lemma has been satisfied and we may conclude that there exists a maximal element M in W. We claim $M = X$. If not then there exists $x \in X$ such that $x \notin M$. Thus we may construct $(\tilde{M}, <) = (M \cup \{x\}, <)$ where $y < x$ for all $y \in M$. $\tilde{M}$ is clearly well ordered and the continuation of M, thus $\tilde{M} > M$. Which is a contradiction since M is maximal. We conclude $M = X$ and thus X is well ordered. ■

Theorem 5.3 *Counting Theorem* Every well ordered set is order isomorphic to a unique ordinal number.

Proof Uniqueness is virtually trivial, since order isomophic is clearly transitive, if a well ordered set were order isomorphic to two different ordinal numbers, those ordinal numbers would be order isomorphic, a contradiction.

Now let X be a well ordered set, let $S = \{x \in X \mid \mathrm{S}(x) \simeq \alpha$ for some ordinal number $\alpha\}$ and let $a \in X$ be an element such that for each predecessor b of a $\mathrm{S}(b)$ is order isomorphic to some ordinal number (clearly the ordinal number is unique and such an element does exist, since the initial segment of the least element of X is the empty set which is order isomorphic to 0, thus the least element of the set $X - \{l \mid l$ is the least element of $X\}$ satisfies the condition for a). Now let $P(x, \alpha)$ be the proposition "α is an ordinal number and $\mathrm{S}(x) \simeq \alpha$". Since α is unique we have $P(x, \alpha)$ and $P(x, \beta) \Rightarrow \alpha = \beta$. We may now apply the Axiom Schema of Replacement and verify the existence of the set $T = \{\alpha \mid x \in \mathrm{S}(a)\ \&\ P(x, \alpha)\}$. T is the set of ordinal numbers order isomorphic to the initial segments determined by the predecessors of a. T is clearly an ordinal number and is order isomorphic to $\mathrm{S}(a)$. We have thus satisfied the hypothesis for transfinite induction and we may conclude that for all $x \in X$ $\mathrm{S}(x) \simeq \alpha$ for some ordinal α. We may annex another element z to X and make it maximal that is $z > x$ for all $x \in X$. Then in the new set $X \cup \{z\}$, $\mathrm{S}(z) = X$, and by the previous argument $X = \mathrm{S}(z) \simeq \alpha$ for some ordinal number α. $\blacksquare$

The Equivalences to the Axiom of Choice

Since every well ordered set is order isomorphic to a unique ordinal number, we present the following definition.

Definition The unique ordinal number to which a well ordered set is order isomorphic is its **order type**, which we shall abbreviate by OT .

We can in fact regard ordinal numbers to be the order types of well ordered sets. However when developing the properties of well ordered sets it is cognitively much easier to work with the narrowly defined ordinal numbers.

Lemma If $A \subset B$ are well ordered sets, then $OT(A) \leq OT(B)$.

Proof Let a and b be ordinal numbers and $\alpha : a \to A$ and $\beta : b \to B$ be order preserving bijections. Either $a \subseteq b$ or $b \subseteq a$. If $a \subseteq b$, then $OT(A) = a \leq b = OT(B)$. If $b \subseteq a$, then let $\phi = \beta^{-1} \circ \alpha$, an order preserving bijection from a to b. Let $c \subseteq a$ such that $\phi(x) = x \ \forall x \in c$. Now let $y \in a$ such that $\mathrm{S}(y) \subset c$. Assume $y \notin c$, then $\phi(y) = z > y$, and $\exists t \in a$ such that $\phi(t) = y$ and $t > y$. But $\phi(t) = y < z = \phi(y)$, which contradicts order preserving. Thus we must have $y \in c$, and by transfinite induction we have $c = a$. Thus $\phi : a \to b$ is the identity map, and since $b \subset a$ we have $b = a$.
■

Corollary If a and b be ordinal numbers that are order isomorphic and $a \subseteq b$, then $a = b$.

If we take the Well ordering theorem as an axiom we may prove the Axiom of Choice as a theorem.

Theorem 5.4 For any non-empty collection of non-empty sets there exists a choice function.

Proof Let C be a non-empty collection of non-empty sets. Well order each member of C and let the choice function choose the least element of each set. ■

We may summarize our results by the following:

Theorem 5.5 The following are equivalent:

1. The Axiom of Choice.
2. Zorn's Lemma.
3. The Well Ordering Theorem.