IX Complex numbers, Quaternions and Octonions

Since the product of two positive real numbers is positive, and the product of any two negative real numbers is also positive, the solution to the equation $x^2 + 1 = 0$ is vacuous in the Real numbers. However by extending the real numbers to a larger set of numbers we can create solutions to equations such as the example given above. The construction of this larger set of numbers from the Real numbers is far easier than the construction of the Reals from the Rationals.

Complex Numbers

Definition The **Complex Numbers**, $\mathbb{C}$, is the cartesian product of the Real Numbers with themselves, $\mathbb{R} \times \mathbb{R}$, with the following arithmetic.

1. $(a, b) + (c, d) = (a + c, b + d)$.
2. $(a, b) \cdot (c, d) = (ac - bd, ad + bc)$.

The real numbers are embedded into the Complex numbers by the following injection map

$$J : x \hookrightarrow (x, 0).$$

Theorem 9.1 The complex numbers are a field.

Proof From the definition of addition and multiplication, the results of the binary operation produces an ordered pair of real numbers hence the Complex numbers are closed with respect to the binary operations of addition and multiplication.

$(a, b) + (c, d) = (a + c, b + d) = (c + a, d + b) = (c, d) + (a, b)$, and $(a, b) \cdot (c, d) = (ac - bd, ad + bc) = (ca - db, cb + da) = (c, d) \cdot (a, b)$. Hence addition and multiplication is commutative.

$((a, b)+(c, d))+(e, f) = (a+c, b+d)+(e, f) = ((a+c)+e, (b+d)+f) = (a + (c + e), b + (d + f)) = (a, b) + (c + e, d + f) = (a, b) + ((c, d) + (e, f)$, and $((a, b) \cdot (c, d)) \cdot (e, f) = (ac - bd, ad + bc) \cdot (e, f) = ((ac - bd)e - (ad + bc)f, (ac-bd)f + (ad+bc)e) = (ace-bde-adf-bcf, acf-bdf+ade+bce) = (a(ce - df) - b(de + cf), a(cf + de) + b(ce - df)) = (a, b) \cdot (ce - df, cf + de) = (a, b) \cdot ((c, d) \cdot (e, f))$. Hence addition and multiplication is associative.

$(a, b) \cdot ((c, d) + (e, f)) = (a, b) \cdot (c+e, d+f) = (ac+ae-bd-bf, ad+af+bc+be) = (ac - bd, ad + bc) + (ae - bf, af + be) = (a, b) \cdot (c, d) + (a, b) \cdot (e, f)$. Hence multiplication distributes over addition.

$(a, b) + (0, 0) = (a + 0, b + 0) = (a, b)$. Thus $(0, 0)$ is the additive identity.

$(a, b) \cdot (1, 0) = (a \cdot 1 - b \cdot 0, a \cdot 0 + b \cdot 1) = (a, b)$. Thus $(1, 0)$ is the multiplicative identity.

$(a, b) + (-a, -b) = (a - a, b - b) = (0, 0)$. Thus any complex number (a, b) has an additive inverse $(-a, -b)$.

$(a, b) \cdot (\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2}) = (\frac{a^2}{a^2 + b^2} - \frac{-b^2}{a^2 + b^2}, \frac{-ab}{a^2 + b^2} + \frac{ab}{a^2 + b^2}) = (\frac{a^2 + b^2}{a^2 + b^2}, \frac{ab - ab}{a^2 + b^2}) = (1, 0)$. Thus for any complex number $(a, b) \neq (0, 0)$ the complex number $(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2})$ is the multiplicative inverse.

We have thus verified that the complex numbers forms a field. ■

We note that the product $(0, 1) \cdot (0, 1) = (-1, 0)$ Since $(-1, 0)$ is the embedding of the real number -1, we have $(0, 1)$ as the square root of -1, and we have a solution to the equation $x^2 + 1 = 0$.

The simplest and standard way to represent Complex numbers is to represent them as the formal expression $a+bi$, where $i^2 = (0,1)(0,1) = (-1,0) = -1$. Now standard arithmetic on binomial expressions yield the appropriate sums and products.

$$(a+bi)+(a'+b'i) = (a+a')+(b+b')i$$

$$(a+bi)(a'+b'i) = aa'+ab'i+a'bi+bb'i^2 = (aa'-bb')+(ab'+a'b)i$$

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For the complex number (a,b) we say $(a,-b)$ is its **complex conjugate**. We denote the complex conjugate of a complex number c by c^*, thus if $c=(a,b)$, then $c^* = (a,b)^* = (a,-b)$.

Definition For $x = (a_1, a_2, \cdots, a_n) \in \mathbb{R}^n = \Pi_{i \in n}\mathbb{R}$, the real value $\sqrt{\sum_{i=1}^{n} a_i^2}$ is the **norm** of x. We say that the norm of a complex number is the norm of the pair of real numbers that represents it.

We see that for any complex number c, we have $c \cdot c^* = (a+bi)(a-bi) = a^2+b^2$ which is the norm squared of c.

Quarternions

Definition The **Quaternions** is the set $\mathbb{C} \times \mathbb{C}$ with the following arithmetic:

1. $(a,b)+(c,d) = (a+c, b+d)$
2. $(a,b)\cdot(c,d) = (ac-d^*b, ad+bc^*)$.

We designate the Quaternions with the blackboard bold face capital H, $\mathbb{H}$.

Let $h \in \mathbb{H}$, then $h = (a + bi, c + di)$. But again it is common practice to write $h = a + bi + cj + dk$.

We leave it to the reader to verify $i^2 = j^2 = k^2 = -1$.

We also leave it to the reader to verify $ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j$. Thus we see $ij \neq ji$, hence quaternions fail to have the commutative property.

Octonions

The **quaternion conjugate** of a quaternion $h = (a, b)$ is $h^* = (a^*, -b)$.

Definition The **Octonions** is the set $\mathbb{H} \times \mathbb{H}$ with the following arithmetic:

1. $(a, b) + (c, d) = (a + c, b + d)$
2. $(a, b) \cdot (c, d) = (ac - d^*b, ad + bc^*)$.

We designate the Octonions with a bold face capital K, $\mathbb{K}$.

Again it is common practice to designate Octonions by $k = a + be_1 + ce_2 + de_3 + ee_4 + fe_5 + ge_6 + he_7$.

We leave it again to the reader to verify $e_n^2 = -1$ and that for $n \neq m$ we have anticommutivity, $e_n e_m = -e_m e_n$.

We could continue constructing numbers in this fashion but more algebraic properties fail. In particular in the next constructions not every non zero number has a multiplicative inverse.