

IV CARDINAL NUMBERS

Cardinality

Definition Let a and b be sets. A subset, m , of $a \times b$ that satisfies the following conditions is called a **bijection**:

- I. $\forall x \in a \exists y \in b$ such that $(x, y) \in m$.
- II. If (x, y) and (x, z) are in m , then $y = z$.
- III. $\forall y \in b \exists x \in a$ such that $(x, y) \in m$.
- IV. If (x, y) and (z, y) are in m , then $x = z$.

Conditions I. and II are the conditions necessary for the subset, m to be a function as defined in Chapter II. We often refer to functions as maps. Thus we say m is a **map** or **function**, and we write $m : a \rightarrow b$.

Condition, III, is called the **onto** condition. Condition, IV, is called the **one to one** condition. Hence we often say a bijection is a map that is one to one and onto.

Definition Let a, b, c , be sets such that there are functions, $f : a \rightarrow b$ and $g : b \rightarrow c$. The **composition** of g with f , written $g \circ f$ is that function that satisfies

$$(g \circ f)(x) = z \iff g(y) = z \text{ where } f(x) = y.$$

Definition Let a and b be sets. If there exists a subset of $a \times b$ that is a bijection then we say a and b are **cardinally equivalent**, or a and b have the same **cardinality**.

Definition A relation on a set a that satisfies the following conditions:

$$\text{R: } xRx \quad \forall x \in a$$

$$\text{S: } xRy \Rightarrow yRx.$$

$$\text{T: } xRy \ \& \ yRz \Rightarrow xRz$$

is called an **equivalence relation**.

Condition R is called reflexive, S is called symmetric, and T is transitive.

We note here that the equivalence relation differs from the order relation by replacing antisymmetry with symmetry.

Definition A **Partition** of a set a , which we shall indicate by $p(a)$, is a subset of the power set of a , $\mathcal{P}(a)$, such that $\bigcup_{p(a)} = a$ and for $b, c \in p(a)$, $b \neq c \Rightarrow b \cap c = \emptyset$.

We leave it as an exercise to the reader to verify that an equivalence relation on a set induces a partition of the set.

We say an equivalence relation partitions a set. That is the set is partitioned into disjoint subsets where each element of each subset is equivalent to each other but not to any element of any other subset.

Theorem 4.1 Cardinal equivalence is an equivalence relation.

Proof Let c be a set.

For reflexivity we note that $\forall a \in c$ the identity map $I : a \rightarrow a$ defined by $I(x) = x$ is a bijection.

For symmetry we note that bijections are one-to-one and onto, thus the reversed relation is also a bijection.

For Transitivity we note that the composition of bijections is also a bijection. ■

Definition A **Cardinal Number** is the least ordinal of that cardinality.

We say a set is finite if it is cardinally equivalent to a proper subset of ω , otherwise we say it is infinite. For finite sets there is a unique ordinal number to which that set is cardinally equivalent, thus for finite sets, ordinal and cardinal numbers are identically the same. This is not true for infinite sets.

For $\omega + 1 = \{0, 1, \dots, \omega\}$ we can form the bijection

$$b : \omega + 1 \leftrightarrow \omega$$

defined by

$$b(x) = x + 1 \text{ for } x \neq \omega \text{ and } b(\omega) = 0.$$

For $\omega + n = \{0, 1, \dots, \omega, \omega + 1, \dots, \omega + (n - 1)\}$ we can form the bijection

$$b : \omega + n \leftrightarrow \omega$$

defined by

$$b(x) = x + n \text{ for } x < \omega \text{ and } b(\omega + k) = k \text{ for } \omega \leq x \leq \omega + (n - 1).$$

Definition Any set that is cardinally equivalent to a subset of ω is said to be **countable**, otherwise we say it is **uncountable**.

Every finite set is cardinally equivalent to a subset of ω , thus all finite sets are countable. If a set is cardinally equivalent to ω then we say the set is **countably infinite**.

Notation: The cardinality of a set, a , is indicated by $C(a)$. The cardinality associated with a countably infinite set is denoted by the cardinal number \aleph_0 (aleph naught, \aleph is the first letter of the Hebrew Alphabet). Thus $C(\omega) = \aleph_0$.

Cantor's Theorem

Theorem 4.2 *Cantor's Theorem* For any set a , $C(\mathcal{P}(a)) \neq C(a)$.

Proof We prove Cantor's theorem by contradiction.

Assume there exists a bijection $b : a \leftrightarrow \mathcal{P}(a)$, and let $y \in \mathcal{P}(a)$ be defined as

$$y = \{x \in a \mid x \notin b(x)\}.$$

The set y exists by the axiom of specification, ZF2. let $c = b^{-1}(y)$. We thus must have the absurd implications

$$c \in y \Rightarrow c \notin b(c) = y \Rightarrow c \in b(c) = y.$$

We conclude no such bijection can possibly exist. ■

Since there is a natural bijection from a set a to the subset of $\mathcal{P}(a)$ that consists of all the singletons it is natural to believe that we in fact have the inequality $C(a) < C(\mathcal{P}(a))$. Since we have defined cardinal numbers in terms of ordinal numbers we wish to delay making this statement until we have demonstrated that every set is cardinally equivalent to some ordinal number.

Corollary There exists an uncountable set.

Proof $\mathcal{P}(\omega)$ is not cardinally equivalent to any subset of ω and thus must be uncountable. ■

The Schröder-Bernstein Theorem

If two sets, A and B , are cardinally equivalent, i.e. a bijection exists between them then we write

$$A \leftrightarrow B.$$

Now suppose that for two sets, A and B , that set B has a subset C , where A is cardinally equivalent to C , ie. $A \leftrightarrow C \subset B$. We then write

$$A \leftrightarrow B.$$

Equivalently we may write

$$B \leftrightarrow A.$$

Theorem 4.3 *Schröder-Bernstein* For any two sets X and Y if $X \hookrightarrow Y$ and $Y \hookrightarrow X$ then $X \leftrightarrow Y$.

Proof By the assumption we have a bijection from X into Y . Call this bijection f , i.e. $f : X \hookrightarrow Y$, or we may write $f : X \leftrightarrow f(X) \subset Y$. We also have by assumption a bijection, g , from Y into X . I.e. $g : Y \hookrightarrow X$. Our goal is to construct a bijection from X to Y . We will proceed as follows. We will partition both X and Y into three disjoint subsets and produce bijections between the subsets of X and Y .

First consider the elements of X that are not in the image of g , i.e., the set $X - g(Y)$. We enlarge this set by including all of its descendants that are in X under the maps $(g \circ f)^n$ and call this set X_X . The set X_X can be specified by

$$X_X = \{z \in X \mid z = (g \circ f)^n(x) \text{ for some } x \in X - g(Y) \text{ and } n \in \omega\}.$$

When $n = 0$ z would be in $X - g(Y)$. We now consider the elements of Y that are descendants of $X - g(Y)$ under the maps $f \circ (g \circ f)^n$. We call this set Y_X . Y_X can be specified by

$$Y_X = \{w \in Y \mid w = f \circ (g \circ f)^n(x) \text{ for some } x \in X - g(Y) \text{ and } n \in \omega\}.$$

We note that $Y_X = f(X_X)$.

We similarly construct the subsets Y_Y and X_Y which can be specified as

$$Y_Y = \{t \in Y \mid t = (f \circ g)^n(y) \text{ for some } y \in Y - f(X) \text{ and } n \in \omega\}.$$

and

$$X_Y = \{u \in X \mid u = g \circ (f \circ g)^n(y) \text{ for some } y \in Y - f(X) \text{ and } n \in \omega\}.$$

Again we note $X_Y = g(Y_Y)$.

We now define the subset X_∞ as those elements of X that are neither in X_X nor X_Y . I.e.

$$X_\infty = \{s \in X \mid s \notin X_X \cup X_Y\}.$$

Similarly we define Y_∞ by

$$Y_\infty = \{r \in Y \mid r \notin Y_Y \cup Y_X\}.$$

We point out here the rationale for the symbols for these sets. X_X is the collection of elements in X whose most distant or primitive ancestor (under the maps $g \circ f$) is in X . X_Y is the collection of elements in X whose most distant ancestor is in Y . X_∞ is the collection of elements of X that have no most distant ancestor, i.e. their lineage can be traced back infinitely far. Similarly for Y_Y , Y_X , and Y_∞ . We also note here that the sets X_X , X_Y and X_∞ are mutually disjoint, as are the sets Y_Y , Y_X and Y_∞ and

$$X = X_X \cup X_Y \cup X_\infty \quad \text{and} \quad Y = Y_Y \cup Y_X \cup Y_\infty.$$

To complete the proof we demonstrate that f restricted to X_X , is a bijection onto Y_X , g restricted to Y_Y is a bijection onto X_Y , and f restricted to X_∞ is a bijection onto Y_∞ . We note here that g restricted to Y_∞ would also be a bijection onto X_∞ . For brevity we will abbreviate a function restricted to a subset of its natural domain by $f|_A$.

We must first show that $f(X_X)$ is in Y_X . Let $z \in X_X$ then $z = (g \circ f)^n(x)$ for some $x \in X - g(Y)$ and $n \in \omega$. Thus $f(z) = f \circ (g \circ f)^n(x)$ for some $x \in X - g(Y)$ and $n \in \omega$. Hence $f(z) \in Y_X$. Clearly $f|_{X_X}$ is one to one since f is one to one. If $y \in Y_X$ then $y = f \circ (g \circ f)^n(x)$ for some $x \in X - g(Y)$ and $n \in \omega$. Thus y is the image under f of an element of the form $(g \circ f)^n(x)$ for some $x \in X - g(Y)$ which is in X_X , hence $f|_{X_X}$ is onto, and thus $f|_{X_X}$ is a bijection.

Similarly $g|_{Y_Y}$ is a bijection from Y_Y to X_Y , and thus $g^{-1}|_{X_Y}$ is a bijection from X_Y to Y_Y .

To demonstrate that $f(X_\infty)$ is in Y_∞ we note that for any $x \in X_\infty$ if $f(x)$ were not in Y_∞ it would be in either Y_Y or Y_X . Thus $f(x)$ would be of the form $(f \circ g)^n(y)$ for some $y \in Y - f(X)$ and $n \in \omega$ or $f \circ (g \circ f)^n(z)$ for some $z \in X - g(Y)$ and $n \in \omega$. Thus $x = f^{-1} \circ (f \circ g)^n(y) = g \circ (f \circ g)^{n-1}(y)$ for some $y \in Y - f(x)$ and $n \in \omega$ or $x = f^{-1} \circ f \circ (g \circ f)^n(z)$ for some $z \in X - g(Y)$ and $n \in \omega$. Thus x would be in either X_Y or X_X which contradicts our assumption, hence $f(X_\infty)$ is in Y_∞ . Again $f|_{X_\infty}$ is one to one since f is one to one. Now let $y \in Y_\infty$ then there exists an $x \in X$ such that $f(x) = y$, if not then $y \in Y - f(X) \subset Y_Y$. If that x were not in X_∞ then it would either be in X_X or X_Y which would mean that y would be in either Y_X or Y_Y and thus not in Y_∞ which contradicts our assumption. Hence we conclude that $f|_{X_\infty}$ is onto and is thus a bijection.

We can now formally define our bijection, $b : X \leftrightarrow Y$, as follows:

$$b(x) = \begin{cases} f(x) & \text{if } x \in X_X \\ f(x) & \text{if } x \in X_\infty \\ g^{-1}(x) & \text{if } x \in X_Y. \quad \blacksquare \end{cases}$$

Exercise In chapter 8 we develop the real numbers, however assuming an *a priori* knowledge of the real numbers consider the closed interval $[0, 1] \equiv X$ and the half-open interval $[0, 1) \equiv Y$. Define injective maps $f : X \hookrightarrow Y$ by $f(x) = \frac{1}{2}x$, and $g : Y \hookrightarrow X$ by $g(x) = x$. Determine the sets $X_X, Y_X, Y_Y, X_Y, X_\infty, Y_\infty$, and the bijection $b : X \leftrightarrow Y$ that the Schroder-Bernstein theorem guarantees to exist.

Some Countable Sets

Theorem 4.4 The finite cartesian product of countable sets is countable.

Proof If the cartesian product of a collection of sets is countable the product can be reindexed to be a single countable set, thus all that needs to be shown is that the cartesian product of two countable sets is countable. this argument is made by the classical diagonalization process.

Let A and B be two countable sets then we can represent the cartesian product by

$$A \times B = \{(a_i, b_j) | a_i \in A, b_j \in B, i, j \in \omega\}$$

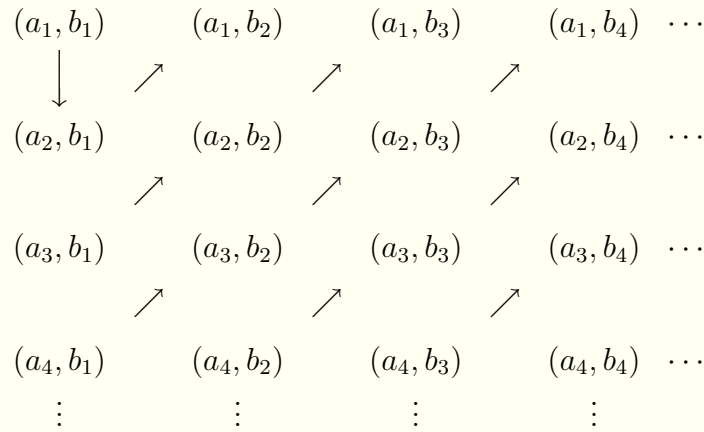
We define a bijection $b : \omega \rightarrow A \times B$ by the following recursion:

$$b : 1 \rightarrow (a_1, b_1)$$

$$b : n \rightarrow (a_{i-1}, b_{j+1}), \text{ where } b : n-1 \rightarrow (a_i, b_j), \text{ where } n > 1, i \neq 1$$

$b : n \rightarrow (a_{j+1}, b_1)$, where $b : n - 1 \rightarrow (a_1, b_j)$, where $n > 1$.

To verify that the map defined by this recursion is a bijection consider the following diagram. The arrows indicate the “order” in which the elements are to be “counted”.



If $b(n) = b(m)$, then the set of predecessors of $b(n)$ is equal to the set of predecessors of $b(m)$. Thus the section of n is equal to the section of m , and thus $n = m$. So b is 1 to 1.

If $(a, b) \in A \times B$, then $(a, b) = (a_k, b_l)$ for some k , and l in ω . Then by “counting” backwards through the recursion we may determine an n where $b(n) = (a, b)$ ■

Corollary $\prod_{i \in n} \omega$ for $n \in \omega$ is countable.

Theorem 4.5 The countable union of countable sets is countable.

Proof Without loss of generality we may assume the sets are disjoint. This is in fact the most extreme case. We may now let a_{ij} represent the i^{th} element of the j^{th} set. The argument is now identical to the proof of theorem 4.4.

■

Corollary ω^ω is countable.

Proof $\omega^\omega = \bigcup_b$ Where $b = \{0, \omega, \omega^2, \dots\}$. ■

From the above arguments it is easily seen that all the ordinal numbers that have explicitly been constructed by the method outlined in chapter III must be countable. Not all ordinals are countable as we shall see in the next chapter.

Some Uncountable Sets

Theorem 4.6 The countably infinite product of countable sets each of which has cardinality of at least 2 is not countable.

Proof Assume the conclusion is false. That is assume there exists a bijection b from ω to $\prod_{i \in \omega} A_i$, where each A_i is countable.

Let p_n be the projection map onto the n^{th} co-ordinate space. For all $n \in \omega$ pick $a_n \in A_n - \{(p_n \circ b)(n)\}$. Then the sequence $(a_0, a_1, \dots) \in \prod_{i \in \omega} A_i$. We have for all $n \in \omega$ $p_n((a_0, a_1, \dots)) = a_n$, but $(p_n \circ b)(n) \neq a_n$. Thus $b(n) \neq (a_0, a_1, \dots) \forall n \in \omega$. Which is a contradiction. Thus no bijection exists. ■

Exercises

The following set of exercises leads to an alternate proof of the Schröder-Bernstein Theorem.

Let X be a partially ordered set and A a subset of X . An element $x \in X$ is said to be an **upper bound** for A if $x \geq a \forall a \in A$. Equivalently, an

element $y \in X$ is said to be a **lower bound** for A if $y \leq a \forall a \in A$. An upper bound u of A is the **Supremum** or **least upper bound** of A if $u \leq x$ for all upper bounds x of A . A lower bound l of A is the **infimum** or **greatest lower bound** of A if $l \geq y$ for all lower bounds y of A .

We say that a partially ordered set X is **complete** if there exists a supremum and infimum for every subset of X .

Let X and Y be partially ordered sets. A function $f : X \rightarrow Y$ is **order preserving** if $x \leq y \Rightarrow f(x) \leq f(y)$.

1. If L is a complete partially ordered set and $f : L \rightarrow L$ is an order preserving function, show that there exists $a \in L$ such that $f(a) = a$.
2. Let A be any set. Show that $\mathcal{P}(A)$ is a complete partially ordered set, where $X \leq Y$ if $X \subseteq Y$ (we say that $\mathcal{P}(A)$ is ordered by inclusion).

For simplicity of notation define $X - A = A'$, that is A' is the complement of A in X .

3. Let A and B be sets and f and g be functions such that $f : A \rightarrow B$ and $g : B \rightarrow A$. Let $h : \mathcal{P}(A) \rightarrow \mathcal{P}(A)$ be defined by $h(S) = [g(f(S)')]'$. Show that $T \subset S \Rightarrow h(T) \subset h(S)$.
4. Observe that h is an order preserving map, thus there exists $S \subset L$ such that $h(S) = S$. that is $g(f(s)')' = X$, and $g(f(S)') = S'$. Now assume f and g are one to one and demonstrate the bijection from A to B .