III ORDINAL NUMBERS

Order Relations

Definition For any set a, each subset of $a \times a$ is called a **relation** on a.

If R is a relation on a set a and if $(x, y) \in R$ then we write xRy.

Definition A relation that satisfies the following conditions:

R $xRx \ \ \forall x \in a$

A xRy & $yRx \ \Rightarrow x = y$.

T xRy & $yRz \ \Rightarrow xRz$

is called an **order relation**, or simply an **order**.

Condition R is called reflexive, A is called antisymmetric, and T is called transitive.

Let a be a set. In our context the elements of a are themselves sets. We see that $R = \{(x, y) \mid x \subset y\}$ is an order relation on a. We leave the verification as an exercise for the reader.

With an order relation we use the symbol $x \preceq y$ instead of xRy. If $x \preceq y$ we say that x precedes y or x is less than or equal to y.

Definition If $x \preceq y$ and $y \not\preceq x$, then we say x is *strictly less than* (or simply *less than*) y, and we write $x \prec y$.

A set together with an order is said to be an **ordered set**. We often refer to an ordered set as a partially ordered set to differentiate it from linearly ordered and well ordered sets which we now define.

Definition An ordered set a that for every $x, y \in a$ either $x \preceq y$ or $y \preceq x$ is said to be **linearly ordered** or **totally ordered**.

Theorem 3.1 *Trichotomy* If a is a linearly ordered set and $x, y \in a$ Exactly one of the following statements is true:
i) $x \prec y$. *ii)* $y \prec x$. *iii)* $x = y$.

Proof If a is linearly ordered then we have

$$x \preceq y \text{ or } y \preceq x \Rightarrow \begin{cases} i) & (x \preceq y \text{ and } y \preceq x) \Rightarrow x = y \text{ or} \\ ii) & (x \preceq y \text{ and } y \npreceq x) \Rightarrow x \prec y \text{ or} \\ iii) & (x \npreceq y \text{ and } y \preceq x) \Rightarrow y \prec x. \end{cases}$$

We observe that neither $x \preceq y$ and $x \npreceq y$, nor $y \preceq x$ and $y \npreceq x$ can be true at the same time, thus the statements are pairwise inconsistent. ■

If $\exists x \in a$ such that $x \preceq y \; \forall y \in a$, then we say x is the **least element** in a.

Definition An ordered set a is said to be **well ordered** if and only if whenever b is any nonempty subset of a, then b has a least element.

Theorem 3.2 Every well ordered set is linearly ordered.

Proof Let a be a well ordered set. Let $x, y \in a$. since a is well ordered the subset $\{x, y\}$ has a least element thus either $x \preceq y$ or $y \preceq x$. Hence a is linearly ordered. ■

Ordinal Numbers

Definition Let a be a set and let $x \in a$, the **section** of x with respect to the set a is

$$\mathrm{S}(x) = \{y \in a \mid y \prec x\}.$$

The **weak section** of x is

$$\bar{\mathrm{S}}(x) = \{y \in a \mid y \preceq x\}.$$

Definition An **ordinal number** is a well ordered set a where for all $x \in a$, $\mathrm{S}(x) = x$.

In the last section we constructed the sets

$$\begin{aligned} \{\} &= 0 \\ \{0\} &= 1 \\ \{0,1\} &= 2 \\ &\vdots \\ \{0,1,\ldots,n-1\} &= n \\ &\vdots \end{aligned}$$

By virtue of the axiom of infinity we may construct the sets

$$\begin{aligned} \omega &= \{0,1,\cdots\} \\ \omega+1 &= \{0,1,\cdots,\omega\} \\ \omega+2 = \omega+1+1 &= \{0,1,\cdots,\omega,\omega+1\} \\ &\vdots \end{aligned}$$

We can easily verify that these sets satisfy the definition of ordinal numbers where the order relation is $x \subset y$.

We now want to construct ‘higher order’ ordinal numbers. The axiom of infinity will not work for us, since it only guarantees the existence of ω. We now must appeal to the axiom schema of replacement (ZF8) to continue our constructions.

Let $P(x, y)$ be the proposition: For $x \in \omega$, y is the x^{th} successor of ω, i.e. $y = \omega + x$. Since ordinal successors are unique if $z = \omega + x$ we must have $y = z$. Thus by the axiom schema of replacement (ZF8) there exists a set b, such that $\omega + n \in b$ for every $n \in \omega$, and conversely b contains only those elements. We see that $b = \{\omega, \omega+1, \omega+2, \cdots\}$. We now construct the union of ω and b.

$$\omega \cup b = \{0, 1, 2, \cdots, \omega, \omega + 1, \omega + 2 \cdots\}$$

We name this set $\omega 2$.

We may repeat this process and construct the set $\omega 3$ where $P(x, y)$ is the proposition: y is the x^{th} successor of $\omega 2$. We continue constructing sets $\omega 4$, $\omega 5, \cdots$. For clarity of discussion we shall refer to ωn as the n^{th} multiple of ω. We let $\omega 0 = 0$ and $\omega 1 = \omega$.

We now let $P(x, y)$ be the proposition: y is the x^{th} multiple of ω. Since successors are unique and multiples are unique collections of successors we have unique multiples. Thus we apply the axiom of replacement (ZF8) and construct the set $b = \{0, \omega, \omega 2, \cdots\}$

We now apply the axiom of unions (ZF4) to construct the union of all sets of b. We designate this set by ω^2.

$$\bigcup_{b} = \omega^2$$

We may visualize ω^2 by the following array.

$$\omega^2 = \left\{ \begin{matrix} 0, & 1, & 2, & \cdots \\ \omega, & \omega + 1, & \omega + 2, & \cdots \\ \omega 2, & \omega 2 + 1 & \omega 2 + 2, & \cdots \\ & \vdots & & \end{matrix} \right\}$$

We note here, that an element of ω^2 is of the form $\omega n + m$ where $n, m \in \omega$.

We may now construct successors of ω^2, $\omega^2 + n$, and by the axiom of replacement form the set $b = \{\omega^2 + n | n \in \omega\}$. The set $\bigcup_b$ we call $\omega^2 + \omega$. We continue as above to form the set $b = \{\omega^2 + \omega n | n \in \omega\}$, and the set $\bigcup_b$ we call $\omega^2 2$. We continue and construct the multiples of ω^2, $\omega^2 n$. Again by virtue of the axiom of replacement we construct a set $b = \{0, \omega^2, \omega^2 2, \omega^2 3, \cdots\}$. And by virtue of the axiom of unions we construct the set $\omega^3 = \bigcup_b$.

We may continue in this fashion constructing the sets ω^4, $\omega^5, \cdots$. We refer to the set ω^n as the n^{th} power of ω. We again apply the axiom of replacement and construct the set

$$b = \{0, \omega, \omega^2, \omega^3, \cdots\}$$

and the union of the sets of b form the set

$$\omega^\omega = \bigcup_b.$$

We observe here that an element of ω^ω can be expressed in the form $\omega^n A_n + \omega^{n-1} A_{n-1} + \cdots + A_0 \equiv \sum_{k=0}^{n} \omega^k A_k$. Where $n \in \omega$ and $A_n \in \omega$.

This process of course does not stop here but continues. We may form sets $\omega^{\omega^\omega}, \cdots$. The set $\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}$ we call ϵ_0. We of course have no reason to believe that we have exhausted all ordinal numbers, and may continue in this fashion ad infinitum.

Transfinite Induction

Theorem 3.3 *The principle of transfinite induction.* If a is an ordinal number and $b \subset a$ such that, for $x \in a$, $\mathrm{S}(x) \subset b \Rightarrow x \in b$, then $b = a$.

Proof Suppose to the contrary that a is an ordinal number and $b \subset a$ such that for $x \in a, \mathrm{S}(x) \subset b \Rightarrow x \in b$ but, there exists $c \in a$ such that $c \notin b$. Then there exists a nonempty set $y = \{x \in a \mid x \notin b\}$. Since y is a nonempty subset of a well ordered set it must have a least element, a_0, and $\mathrm{S}(a_0) \subset b$, thus by our hypothesis $a_0 \in b$ which contradicts our assumption. Thus we must conclude $b = a$. ■

Properties of Ordinal Numbers

Lemma 3.4 The elements of ordinal numbers are ordinal numbers.

Proof If a is an ordinal number and $b \in a$, then $b = \mathrm{S}(b) \subset a$. We notice that any subset of b is a subset of a, thus b is well ordered, furthermore for any $c \in b$ we have $c \in a$ and thus $c = \mathrm{S}(c)$. Therefore b is an ordinal number. ■

Theorem 3.5 Let a, b be ordinal numbers, then either $a \subsetneq b$, or $b \subsetneq a$, or $a = b$.

Proof Either $a = b$, or $a \neq b$. If $a \neq b$, then either $\exists x \in a$ where $x \notin b$ or $\exists x \in b$ where $x \notin a$. If $\exists x \in a$ where $x \notin b$, let $t = \{x \in b | x \in a\} \subset b \cap a \subset a$. We show $t = b$ by transfinite induction, and hence $b \subset a$. Let $x \in b$ such that $\mathrm{S}(x) \subset t$, thus $\mathrm{S}(x) \subsetneq a$. Since $\mathrm{S}(x)$ is a proper subset of a we have $\{y \in a | y \notin \mathrm{S}(x)\} \neq \emptyset$. Let r be the least element of $\{y \in a | y \notin \mathrm{S}(x)\}$, then $r = \mathrm{S}(x) = x$. Since $r \in a$ we have $x \in a$ and thus $x \in t$. Hence by Transfinite induction $t = b$. By the symmetric argument if $\exists x \in b$ where $x \notin a$ we have $a \subset b$. ■

Definition An **upper bound** for an ordered set $\mathcal{C}$ is an element β such that $x \preceq \beta \ \forall x \in \mathcal{C}$.

Definition A **supremum** or **least upper bound** for an ordered set $\mathcal{C}$ is an element α such that α is an upper bound, and if γ is an upper bound, then $\alpha \preceq \gamma$. We indicate the supremum of an ordered set $\mathcal{C}$ by $\sup \mathcal{C}$.

When the elements of an ordered sets are regarded as numbers we will use the symbols $\leq$ and $<$ for $\preceq$ and $\prec$. Thus for ordinal numbers the symbols $\leq$, $\preceq$ and $\subset$ are equivalent, as are $<$, $\prec$ and $\subsetneq$.

Theorem 3.6 If $\mathcal{C}$ is a set of ordinal numbers, then $\mathcal{C}$ has a supremum.

Proof Let $\alpha = \bigcup\limits_{\mathcal{C}}$. We claim that α is an ordinal number. Let $A \subset \alpha$ and $A \neq \emptyset$. Pick $a \in A$, if $a \leq b \ \forall b \in A$, then a is the least element. If a is not the least element, then $\exists\ b \in A$ such that $b < a \Rightarrow b \in a$. Thus $a \cap A \neq \emptyset$. The element a is an ordinal number and is well ordered. Let a_0 be the least element of $a \cap A$. Let c be an arbitrary element in A, then either $a \leq c$ or $c < a$. If $a \leq c$, then $a_0 \leq c$. If $c < a$, then $c \in a \cap A$, and thus $a_0 \leq c$. Thus a_0 is the least element of A. If $\xi \in \alpha$, then $\xi \in c$ for some $c \in \mathcal{C} \Rightarrow \ \xi = \mathrm{S}(\xi)$. Thus α is an ordinal number. Now α is an upper bound for $\mathcal{C}$, since if $c \in \mathcal{C}$, then $c \subset \alpha$ implies $c \in \alpha$. Now suppose ζ is an upper bound for $\mathcal{C}$. Then for all $c \in \mathcal{C}$ we have $c \subset \zeta$, and thus $\alpha \subset \zeta \ \Rightarrow \ \alpha < \zeta$. Thus α is the supremum. ■

Corollary The collection of all ordinal numbers is a proper class.

Proof If the collection were a set, then a supremum would exist. Let α be the supremum, but $\alpha \subset \alpha + 1$ and $\alpha + 1$ is an ordinal number. Thus $\alpha + 1 \in \alpha \in \alpha + 1$. Which contradicts Lemma 2.7. ■

As promised at the end of chapter II, we now state and prove a generalization of the final two lemmas of that chapter. As you will note we need the concept of ordinal numbers to state the theorem in its generality.

Definition An ordinal number greater than 0 that is not the successor of any other ordinal number is said to be a **limit ordinal**.

Theorem 3.7 For any collection of sets C, that can be indexed by an ordinal $\alpha+1$, that is not a limit ordinal, we can never have, for $C = \{x_\lambda | \lambda \in \alpha+1\}$, $x_0 \in x_1 \in \cdots \in x_\alpha \in x_0$.

Proof If we had $x_0 \in x_1 \in \cdots \in x_\alpha \in x_0$, then we would always have for every $\nu \neq \alpha$, $x_\nu \in C \cap x_{\nu+1}$ and $x_\alpha \in C \cap x_0$. Which contradicts the axiom of regularity (ZF9). ■

Corollary For any two sets a and b, $a \cap (b \times \{a\}) = \emptyset$.

Proof Every element of $b \times \{a\}$ is of the form $\{\{b'\}, \{b', a\}\}$ which cannot be in a, by virtue of theorem 3.7. ■

The Transfinite Recursion Theorem.

Let W be a well-ordered set and $\alpha \in W$. An **α-sequence** in a set X is a function $\phi : \mathrm{S}(\alpha) \to X$. Recall that $\mathrm{S}(\alpha)$ is the initial section of α.

A **sequence function of type** W in X is a function

$$f : \{\phi : \mathrm{S}(\alpha) \to X | \alpha \in W\} \to X.$$

That is, f maps α-sequences into X.

Let $\Upsilon : W \to X$ where W is a well ordered set and X is a set. We observe that $\Upsilon|_{\mathrm{S}(\alpha)} : \mathrm{S}(\alpha) \to X$ is an α-sequence for all $\alpha \in W$. $\Upsilon|_{\mathrm{S}(\alpha)}$ is the restriction of Υ to $\mathrm{S}(\alpha)$).

Theorem 3.8 *Transfinite Recursion Theorem* If W is a well ordered set and if f is a sequence function of type W in a set X, then there exists a unique function $\Upsilon : W \to X$ such that $\Upsilon(\alpha) = f(\Upsilon|_{\mathrm{S}(\alpha)})$ for each $\alpha \in W$.

Proof To prove uniqueness, let Υ and Ψ be two such functions such that $\Upsilon(\beta) = \Psi(\beta)\ \forall \beta \in \mathbf{S}(\alpha)$. That is $\Upsilon|_{\mathrm{S}(\alpha)} = \Psi|_{\mathrm{S}(\alpha)}$. Then we have

$$\Upsilon(\alpha) = f(\Upsilon|_{\mathrm{S}(\alpha)}) = f(\Psi|_{\mathrm{S}(\alpha)}) = \Psi(\alpha).$$

Thus by Transfinite induction we have $\Upsilon(\alpha) = \Psi(\alpha)\ \forall \alpha \in W$.

To prove existence we explicitly construct Υ as a subset of $W \times X$.

We say a subset A of $W \times X$ is **f-closed** if for $\alpha \in W$ and t an α-sequence in A, i.e. $\{(c, t(c)) | c \in \mathrm{S}(\alpha)\} \subset A$, then $(\alpha, f(t)) \in A$. $W \times X$ is f-closed, thus such subsets do exist.

Let $\Upsilon = \bigcap\limits_{A \text{ is } f-\text{closed}} A$. Υ is f-closed since any α-sequence is in every A. Thus $(\alpha, f(t)) \in A$ for every A, and $(\alpha, f(t)) \in \Upsilon$.

We now show that Υ is a function. That is $\forall \gamma \in W\ \exists\,!\ \xi \in X$ such that $(\gamma, \xi) \in \Upsilon$.

We proceed by transfinite induction on W.

Let $S = \{\gamma \in W | (\gamma, \xi), (\gamma, \zeta) \in \Upsilon\ \&\ (\gamma, \xi) = (\gamma, \zeta) \Rightarrow \xi = \zeta\}$. Also let $\mathrm{S}(\alpha) \subset S$ for some α. Thus if $\gamma < \alpha$, then $\exists\,!\ \xi \in X$ such that $(\gamma, \xi) \in \Upsilon$. The function $t : \mathrm{S}(\alpha) \to \xi$, thus defined, is an α-sequence and $t \subset \Upsilon$.

Now assume $\alpha \notin S$. Then $(\alpha, y) \in \Upsilon$ where $y \neq f(t)$. Now consider the set $\Upsilon - \{(\alpha, y)\}$. Let $\beta \in W$ and r be a β-sequence in $\Upsilon - \{(\alpha, y)\}$. If $\beta = \alpha$, then $r = t$ by the uniqueness of Υ. Also $(\beta, f(r)) = (\alpha, f(t)) \in \Upsilon - \{(\alpha, y)\}$, since $(\alpha, f(t)) \neq (\alpha, y))$ and $(\alpha, f(t)) \in \Upsilon$. If $\beta \neq \alpha$, we have $(\beta, f(r)) \in \Upsilon - \{(\alpha, y)\}$ since Υ is f-closed and $(\beta, f(r)) \neq (\alpha, y)$. Thus we have, if $\beta \in W$ and if r is a β-sequence in $\Upsilon - \{(\alpha, y)\}$, then $(\beta, f(r)) \in \Upsilon - \{(\alpha, y)\}$. That is to say $\Upsilon - \{(\alpha, y)\}$ is f-closed. This contradicts the fact that Υ is the smallest f-closed set. We must conclude that $\alpha \in S$. The hypothesis for transfinite induction has been verified. Thus the existence of Υ has been demonstrated. $\blacksquare$