II THE AXIOMS

The classical collection of Axioms for set theory are the Zermelo Fraenkel Axioms (ZF Axioms).

The Axiom of Extension

ZF1 *Axiom of extension.* Two sets are equal if and only if they have identically the same elements. $\forall a \forall b (a = b) \Leftrightarrow \forall x (x \in a \Leftrightarrow x \in b)$.

This axiom not only defines the term *equal* and its associated symbol, $=$, but also tells us that it is the elements that uniquely define a set. The set may however have different descriptions. For example consider those planets that orbit the sun closer than the earth, and those planets of our solar system with no natural satellite. A simple check of an almanac reveals that exactly the same collection of planets satisfies both descriptions.

Let a be any set. Let $P(x)$ be a proposition about an arbitrary element x from a; that is, for every $x \in a$, $P(x)$ is a statement that is either true or false. An example of a non-proposition in the variable x is $x \in b$. In this example b is also an indeterminate. Without specifying b we have no way of knowing whether any particular element of our initial set a is in b or not.

The Axiom Schema of Specification

ZF2 *Axiom schema of specification.* For every set a and proposition $P(x)$ there is a set b that consists of those elements of a where $P(x)$ is true.
$\forall a \exists b (x \in b \Leftrightarrow x \in a \wedge P(x))$.

ZF2 is not regarded as a single axiom but rather as a collection of axioms, hence the term axiom schema is used. Each possible proposition $P(x)$

produces a separate axiom. Several of the Zermelo Fraenkel Axioms are in fact axiom schema and will be identified as such.

By convention, braces { and } are used when representing a set, with the elements listed between the braces. The pair of braces should be considered as a single symbol, a single brace without its complement is meaningless in set theory. The braces can be thought of as the purse that holds the coins. The purse is by no means a member of the collection of coins, and the braces are not elements of a set. The braces simple mean that the elements listed between them are to be considered a set. $\{x, y, z\}$ is the set that consists of the letters x, y, z (the commas, or course, are punctuation).

When we use an axiom of specification we express the set b (of ZF2) by $\{x \in a \mid P(x)\}$, which is read: The set of all x in a such that $P(x)$. The set a is called the Universal Set and $P(x)$ is the proposition that must be satisfied for an element to be included in the set b.

Let a be an arbitrary set, for example let a be the collection of nations in the United Nations. Now let $P(x)$ be the statement: x is not x, e.g., Canada is not Canada. At first glance this statement may seem absurd, but in reality it is not absurd but simply false for every element of our set. Using our notation we write the set specified as:

$$\{x \in a \mid x \neq x\}.$$

By virtue of the axiom schema of specification this set exists, but no element of a satisfies the proposition $x \neq x$. Hence we must conclude that the set is vacuous and we call it the Empty or Null Set. The empty set is not just an interesting or pathological footnote to the axiom of specification, but is absolutely crucial to the study of sets. In fact we may at this time choose to disregard any other set (and I choose to do so) and focus our attention solely on the empty set and other sets that may be derived from it by the axioms. We reserve the symbols $\emptyset$ and $\{\ \}$ for the empty set. We may state this result as a theorem.

Theorem 2.1 There exists a set that has no elements.

Here is the formal proof.

Proof Let $\emptyset = \{x \in a \mid x \neq x\}$. Since for any set a no element satisfies the proposition $x \neq x$, we conclude $\emptyset$ has no elements. ■

An interesting result known as *Russell's Paradox* can now be presented. Since sets themselves can be regarded as objects, it seems quite reasonable to consider a collection of sets as a set itself. Let s be the collection of all sets and consider the following set.

$$r = \{x \in s \mid x \notin x\}.$$

Now the set r is either an element of itself or it is not. Assume that it is, i.e. we assume $r \in r$. Then by the defining proposition of the set r we see that it must be the case that $r \notin r$, contrary to the assumption. Thus we may assume that the other case is true, that r is not an element of itself, i.e $r \notin r$, but we see that if that is true we must necessarily have $r \in r$. Again this presents a contradiction. Clearly the collection, s, is much too large or encompassing to be considered a set. This unfortunate result is handled by an appropriate axiom yet to be stated. (See ZF9)

At this stage the only sets that we have are real, tangible objects and the empty set. We wish to present axioms that allow us to expand our collection of sets to more abstract objects. The next few axioms allow us to construct new sets from preexisting sets.

The Axiom of Pairing

ZF3 *Axiom of pairing.* If a and b are sets then there exists a set c such that $a \in c$ and $b \in c$. $\forall a \forall b \exists c (a \in c \wedge b \in c)$.

There is an important fact to note here. The axiom specifies two elements that are in c but says nothing about what else may be in c, in this sense c is not a well defined set. But by the use of an axiom of specification we may construct a well defined set. Let c be a set guaranteed to exist by the axiom of pairing and consider the following set:

$$d = \{x \in c \mid x = a, \text{ or } x = b\}.$$

Clearly $d = \{a, b\}$.

Another point to note here is that a and b may be the same set, that is $a = b$. In this case we have the following:

$$d = \{a, b\} = \{a, a\} = \{a\}$$

The last equation is true by virtue of the axiom of extension. Since both sets on each side of the equal sign have exactly the same elements the statement is true.

Using only the empty set, we can now construct a new set. Let $a = b = \emptyset$, and we have $d = \{\emptyset\}$.

We may continue ad infinitum constructing new sets from previously constructed sets, for example let $a = \emptyset$ and $b = \{\emptyset\}$, and we now have

$$d' = \{\emptyset, \{\emptyset\}\}.$$

The Axiom of Unions

ZF4 *Axiom of unions.* For every set c there is a set a where, if $b \in c$ and $x \in b$, then $x \in a$. $\forall c \exists a \forall b (x \in b \wedge b \in c \Rightarrow x \in a)$.

Here we regard the set c as a collection of sets. We form a new set by including every element of each of the member sets of c. If c is not a collection of sets then the statement is vacuously true and a is possibly empty. Again a is not a well defined set, the axiom does not exclude other possible elements that do not satisfy our stated condition. An application of the axiom of specification (ZF2) will form a set that contains only those elements that are contained in the sets of c.

Let a' represent that set whose elements consist of only those elements of a that are elements of members of c. We write

$$a' = \bigcup_{b \in c} b \quad \text{or simply} \quad a' = \bigcup_{c}$$

which is read a prime is the **union** of c.

If c consists of only two elements, $b_1,\ b_2$, then we write $b_1 \cup b_2$.

A closely related concept to the union of sets is the intersection of sets. Let c be a collection of sets, we define the intersection of this collection of sets by

$$\bigcap_{b \in c} b = \{x \in \bigcup_{c} \mid x \in b \text{ for every } b \text{ in } c\}.$$

The reader may question why we choose the Union of c for our Universal set when any element of c would suffice. By choosing the Union we eliminate the problem of devising some mechanism to choose some element of c to be our Universal set.

If we are constructing the intersection of only two sets, $b_1,\ b_2$, then we write $b_1 \cap b_2$.

Definition We say the sets a and b are **disjoint**, if $a \cap b = \emptyset$.

Definition For two sets a and b we say a is a **subset** of b, written $a \subset b$, if and only if every element of a is an element of b; i.e. $a \subset b \ \Leftrightarrow \ (x \in a \ \Rightarrow \ x \in b)$.

Theorem 2.2 $a = b$ if and only if $a \subset b$ and $b \subset a$.

This theorem follows immediately from the definition of subset and the axiom of extension (ZF1).

Note that it is vacuously true that $\emptyset \subset a$ for every set a.

If $a \subset b$ and $a \neq b$, we say that a is a **proper subset** of b, and we write $a \subsetneq b$. **Definition** For sets a and b, the **complement** of b in a, written $a - b$ is the set

$$a - b = \{x \in a \mid x \notin b\}.$$

Theorem 2.3 *De Morgan laws* If c is a set and b is a collection of subsets of c, then $c - \bigcup_{a \in b} a = \bigcap_{a \in b} (c - a)$ and $c - \bigcap_{a \in b} a = \bigcup_{a \in b} (c - a)$.

Proof

$$\begin{aligned}
\Big[x \in c - \bigcup_{a \in b} a \quad &\Rightarrow \quad x \in c \text{ and } x \notin \bigcup_{a \in b} a \\
&\Rightarrow \quad x \in c \text{ and } \forall a \in b \ x \notin a \\
&\Rightarrow \quad \forall a \in b \ x \in (c - a) \\
&\Rightarrow \quad x \in \bigcap_{a \in b} (c - a)\Big] \\
&\Rightarrow \quad c - \bigcup_{a \in b} a \subset \bigcap_{a \in b} (c - a)
\end{aligned}$$

On the other hand

$$
\begin{aligned}
\Big[x \in \bigcap_{a\in b}(c-a) \quad &\Rightarrow \quad \forall\ a \in b\ x \in (c-a) \\
&\Rightarrow \quad x \in c \text{ and } \forall\ a \in b\ x \notin a \\
&\Rightarrow \quad x \in c \text{ and } x \notin \bigcup_{a\in b} a \\
&\Rightarrow \quad x \in c - \bigcup_{a\in b} a\Big] \\
&\Rightarrow \quad \bigcap_{a\in b}(c-a) \subset c - \bigcup_{a\in b} a
\end{aligned}
$$

By Theorem 2.2 we have the result

$$c - \bigcup_{a\in b} a = \bigcap_{a\in b}(c-a).$$

The proof that

$$c - \bigcap_{a\in b} a = \bigcup_{a\in b}(c-a)$$

is similar and left to the reader. ■

The Axiom of Power Sets

ZF5 *Axiom of power sets.* For any set a there is a set b such that if $x \subset a$, then $x \in b$. $\forall a \exists b (x \subset a \Rightarrow x \in b)$.

Again we must apply an axiom of specification (ZF2)to construct a well defined set whose elements are only the subsets of a. This set is called the **power set** of a and is written $\mathcal{P}(a)$.

By virtue of the axiom of pairing (ZF3) for any set a we may construct the singleton set $\{a\}$. Now by virtue of the axiom of unions(ZF4) we may construct the set $a \cup \{a\}$. This set is known as the successor of a and is written $a+1$. We formally define the successor by:

Definition For any set a the **successor** of a is the set $a+1$ given by:

$$a + 1 = a \cup \{a\}.$$

Examples:

$$\emptyset + 1 = \emptyset \cup \{\emptyset\} = \{\emptyset\}$$

$$\emptyset + 1 + 1 = \{\emptyset\} \cup \{\{\emptyset\}\} = \{\emptyset, \{\emptyset\}\}$$

For convenience we name these sets, $\emptyset = 0$, $\{\emptyset\} = 1$, $\{\emptyset, \{\emptyset\}\} = 2$.

Thus we see:

$$\begin{aligned} \{\ \} &= 0 \\ \{0\} &= 1 \\ \{0, 1\} &= 2 \\ &\vdots \\ \{0, 1, \ldots, n-1\} &= n \\ &\vdots \end{aligned}$$

Remark: The notation $a + 1 + 1 + \cdots + 1$ is unambiguous as its only interpretation is $(\ldots((a+1)+1)+\cdots+1)$. To associate differently, e.g. $a + (1 + 1 + \cdots + 1)$ has no meaning. The notation $a + n$ refers to the n^{th} successor of a. That is $a + n = a + \underbrace{1 + 1 + \cdots + 1}_{n}$.

The Axiom of Infinity

The following axiom is a powerful statement that allows for the construction of arbitrarily large sets and allows us to regard unbounded classes of numbers as sets.

ZF6 *Axiom of infinity.* There exists a set, a, that contains $\emptyset$, and the successor of each of its elements. $\exists a(\emptyset \in a \wedge (x \in a \Rightarrow x+1 \in a)$.

A set that satisfies the axiom of infinity is called a **successor set**.

Theorem 2.4 If a and b are successor sets, then $a \cap b$ is a successor set.

Proof $(a,\ b$ successor sets$) \Rightarrow (\emptyset \in a$ and $\emptyset \in b) \Rightarrow (\emptyset \in a \cap b)$. $(\alpha \in a \cap b) \Rightarrow (\alpha \in a$ and $\alpha \in b) \Rightarrow \alpha + 1 \in a$ and $\alpha + 1 \in b) \Rightarrow \alpha + 1 \in a \cap b)$.
■

We generalize this theorem to include arbitrary intersections and hence the above theorem becomes a corollary to the following theorem.

Theorem 2.5 If A is an arbitrary collection of successor sets, then $\bigcap_A$ is a successor set.

Proof Let A be a collection of successor sets. $\emptyset \in a\ \forall a \in A \ \Rightarrow\ \emptyset \in \bigcap_{a \in A} a$.
$\alpha \in \bigcap_{a \in A} a \ \Rightarrow\ \alpha \in a\ \forall a \in A \ \Rightarrow\ \alpha + 1 \in a\ \forall a \in A \ \Rightarrow\ \alpha + 1 \in \bigcap_{a \in b} a$. ■

Let Ω be a successor set, and let $A = \{a \in \mathcal{P}(\Omega) | a$ is a successor set $\}$, that is A is the collection of all successor subsets of Ω. We let $\omega = \bigcap_A$.

We notice ω is unique regardless of the initial choice of Ω since if we let Ω_1 and Ω_2 be two successor sets we have $\Omega_1 \cap \Omega_2$ is a successor set and $\Omega_1 \cap \Omega_2 \subset \Omega_1$ and $\Omega_1 \cap \Omega_2 \subset \Omega_2$.

We also see ω is the minimal successor set. Thus we see $\omega = \{0, 1, 2, \dots\}$.

We can now construct the successors of ω, $\omega + 1 = \{0, 1, 2, \dots, \omega\}$ and $\omega + 1 + 1 = \{0, 1, 2, \dots, \omega, \omega + 1\}$

We name $\omega + 1 + 1 = \omega + 2$ and $\omega + \underbrace{1 + \cdots + 1}_{n} = \omega + n$.

It is important to observe here that the successor of a set is **not** a successor set.

The Ordering of Sets, Cartesian Products and Functions

Definition We say that a **is less than** b, written $a < b$, iff $a \in b$.

We should note here that sets and elements are somewhat synonymous, they differ only in their relationship to each other.

Now we wish to deal with the situation where we have a set with two or more elements and we wish to designate one element as the first element, and another as the second and so forth. Let us begin with the easiest case, a set with two elements.

Let x and y be elements, by virtue of the axiom of pairing (ZF3) we can construct the two sets $\{x, y\}$, and $\{x\}$. Again using the axiom of pairing we construct the set $\{\{x\}, \{x, y\}\}$. For simplicity of notation we use (x, y) to represent the set $\{\{x\}, \{x, y\}\}$.

Definition The **ordered pair**, (x, y), is the set $\{\{x\}, \{x, y\}\}$.

We regard the ordered triple (x, y, z) as the ordered pair

$$((x, y), z) = \{\{\{\{x\}, \{x, y\}\}\}, \{\{\{x\}, \{x, y\}\}, z\}\}.$$

Thus we may inductively define the ordered n-tuple by

$$(x_1, x_2, \cdots, x_n) = ((x_1, x_2, \cdots, x_{n-1}), x_n).$$

We now want to consider certain subsets of the collection of all ordered pairs formed from two sets.

Definition The **Cartesian product** of two sets a and b is

$$a \times b = \{z \in \mathcal{P}(\mathcal{P}(a \cup b)) \mid z \; = \{\{x\}, \{x, y\}\} \text{ where } x \in a,\; y \in b\}.$$

For simplicity of notation we write

$$a \times b = \{(x, y) | x \in a,\; y \in b\}.$$

Definition A **function** from a set a to a set b is a subset, f, of $a \times b$ that satisfies the following two condition:

1. $\forall x \in a \; \exists (x, y) \in f$, and
2. if $(x, y) \in f$ and $(x, z) \in f$ then $y = z$.

To work with functions efficiently, it helps to name the related sets. The set a is called the **domain** of f, and b is called the **codomain** of f. The **range** of f is the set $\{y \in b | (x, y) \in f\}$. If $c \subset a$, then the **image** of c under f is the set $\{y \in b \mid x \in c$ and $(x, y) \in f\}$. If $d \subset b$, then the **preimage** of d under f is the set $\{x \in a | (x, y) \in f$ and $y \in d\}$. We write $f(c)$ for the image of c under f, and $f^{-1}(d)$ for the preimage of d under f. Also, if the range of a function f is equal to its codomain, we say the function is **onto** its codomain.

We express a function, f, with domain a and codomain b by $f : a \to b$. Also for a function f, if $(x, y) \in f$, then we express y as $f(x)$. We may write $y = f(x)$, or $(x, f(x))$ to represent the element (x, y) of f.

We wish to generalize the cartesian product to arbitrary collections of sets. To do so we introduce the concept of indexing one set by another.

Definition A function I from a set Λ onto a set a is said to **index** the set a by Λ. The set Λ is called the **index** and a is the indexed set. If $I(\lambda) = a$, then we write a_λ for $I(\lambda)$.

Definition Let a be a non-empty set (we remind the reader here, that the elements of sets are considered to be sets) indexed by a set Λ. The **cartesian product** of a is defined to be the collection, Π, of all functions with domain Λ and codomain $\bigcup\limits_{b \in a} b$, satisfying the condition $f(\lambda) \in b_\lambda$. We write

$$c = \prod_{\lambda \in \Lambda} b_\lambda.$$

For clarity we present some examples here.

Let $a = \{\{1, 2\}, \{2, 3\}\}$. Then we have the four functions

$$f_1 = \{(\{1, 2\}, 1), (\{2, 3\}, 2)\}$$

$$f_2 = \{(\{1, 2\}, 1), (\{2, 3\}, 3)\}$$

$$f_3 = \{(\{1, 2\}, 2), (\{2, 3\}, 2)\}$$

$$f_4 = \{(\{1, 2\}, 2), (\{2, 3\}, 3)\}$$

This notation is cumbersome but we can represent these functions by the following ordered pairs without loss of any information.

$$f_1 = (1, 2), \quad f_2 = (1, 3), \quad f_3 = (2, 2), \quad f_4 = (2, 3).$$

Hence the cartesian product is

$$\prod = \{(1,2),(1,3),(2,2),(2,3)\}.$$

In this example where there are only two elements in a we may write

$$\{1,2\} \times \{2,3\} = \{(1,2),(1,3),(2,2),(2,3)\}.$$

We leave as an exercise for the reader to demonstrate that the cartesian product of the following collection

$$a = \{\{1,2\},\{2,3\},\{3,4\}\}$$

can be represented by

$$\prod_{\lambda\in 3} \alpha_\lambda = \{(1,2,3),(1,2,4),(1,3,3),(1,3,4),(2,2,3),(2,2,4),(2,3,3),(2,3,4)\}.$$

Let $a = \{\{0,1\}\}$ be indexed by ω. The cartesian product of a with respect to this indexing can be represented by the set of infinite strings of zeros and ones. That is

$$\prod_{i\in\omega}\{0,1\}_i = \{(b_0, b_1, \cdots)|b_i = 0 \text{ or } 1\ \forall i \in \omega\}.$$

Any function, f, that satisfies the conditions in the previous definition is called a **choice function**. The rationale for this name is clear, as the function chooses an element from each set.

Let a be a set and $c = \prod_{\lambda\in\Lambda} b_\lambda$. The **projection map** $p_{b_\lambda} : c \to b_\lambda$ is the function defined by $p_{b_\lambda}(x) = x(\lambda)$.

For our original example we may compute

$$p_{\{1,2\}}(f_1) = f_1(\{1,2\}) = 1$$

and

$$p_{\{1,2\}}(f_3) = f_3(\{1,2\}) = 2.$$

We leave as an exercise to compute $p_{\{1,2\}}(f_2)$ and $p_{\{1,2\}}(f_4)$.

The Axiom of Choice

The next axiom is known as the Axiom of Choice. We give three formulations of the statement of this axiom.

ZF7 *Axiom of Choice.*

I. For every nonempty set whose elements are nonempty sets there exists a choice function.

II. If $\{a_i\}$ is a family of nonempty sets, indexed by a nonempty set I, then there exists a family $\{x_i\}$ with $i \in I$ such that $x_i \in a_i$ for each $i \in I$.

III. The cartesian product of a nonempty collection of nonempty sets is nonempty. $(\forall a \neq \emptyset \vee \; x \in a \Rightarrow x \neq \emptyset) \Rightarrow \prod_{x \in a} \neq \emptyset$.

Theorem 2.6 The three previous statements are equivalent.

Proof I. $\Rightarrow$ II. Let $\mathbf{A}$ be a collection of disjoint nonempty sets. We have $\mathbf{A} \subset \mathcal{P}(\bigcup_{i \in I} a_i)$. By I. there exists a choice function f on $\mathcal{P}(\bigcup_{i \in I} a_i)$. Let b be the image of $\mathbf{A}$. Pick an element $a \in \mathbf{A}$, $f(a) \in a \cap b$ since $f(a) \in a$. Let $y \in b$ where $y \neq f(a)$ thus we have $y = f(a')$ where $a' \neq a$, and thus $y \in a'$. Since a and a' are disjoint $y \notin a$. Thus the only element of $b \cap a$ is $f(a)$.

II. $\Rightarrow$ I. Define the choice function to be $\{(a_i, x_i) \mid i \in I\}$.

I $\iff$ III. Since the cartesian product is the collection of choice functions, if a choice function exists then the cartesian product is non-empty. Conversely if the cartesian product is non-empty its elements are choice functions, thus a choice function exists. ■

The Axiom Schema of Replacement

ZF8 *Axiom schema of replacement.* If $P(x, y)$ is a proposition such that for each x in a set a, $P(x, y)$ and $P(x, z)$ implies that $y = z$, then there exists a set b such that $y \in b$ if and only if there exists an x in a such that $P(x, y)$. $\forall x \in a(P(x, y) \wedge P(x, z) \Rightarrow y = z) \Rightarrow (\exists b \wedge (y \in b \Leftrightarrow \exists x \in a \wedge P(x, y)))$.

ZF8 allows the construction of new sets in the following way. If a set a exists and a rule that assigns to elements of a other pre-existing elements or sets that may or may not be elements of other sets, then there exists a set b that contains only those elements. This axiom schema will be heavily relied upon in the next chapter.

The Axiom Schema of Restriction

ZF9 *Axiom schema of restriction.* Let S(x) be any proposition involving x that does not involve y or z. If there exists an x such that $S(x)$ is true, then there exists a y such that $S(y)$ is true and, for all z, if $z \in y$ then $S(z)$ is false.

If we take $S(x)$ to be the statement $x \in a$, then we have the following statement, which we call the axiom of regularity.

Axiom of regularity. Every nonempty set a contains an element b such that $a \cap b = \emptyset$. $\forall a \neq \emptyset \exists b \in a \wedge a \cap b = \emptyset$.

Two important lemmas follow from the axiom of regularity.

Lemma 2.7 For each set $a, a \notin a$.

Proof The proof is indirect. We assume that there exists a set a such that $a \in a$. We thus have $a \in \{a\} \cap a$. However by the axiom of regularity $\{a\}$ contains an element whose intersection with $\{a\}$ is empty. Since a is the only element we have $\{a\} \cap a = \emptyset$, which contradicts our original assumption. ■

Corollary There does not exist a set of all sets.

Proof If there existed a set of all sets it would have to be an element of itself which would contradict the previous lemma. ■.

Thus we see the axiom of regularity is the response to Russell's paradox.

Lemma 2.8 No two sets can be elements of each other.

Proof Again the proof is indirect. Assume that a and b are sets such that $a \in b$ and $b \in a$. We thus have $a \in \{a, b\} \cap b$ and $b \in \{a, b\} \cap a$. By the Axiom of regularity we must have an element $x \in \{a, b\}$ such that $x \cap \{a, b\} = \emptyset$. Since our only two choices are a or b we must have either $\{a, b\} \cap a = \emptyset$ or $\{a, b\} \cap b = \emptyset$ which contradicts our assumption. ■

These two lemmas can be replaced by a more general theorem (Theorem 3.3) that will be stated and proved in the next chapter. Meanwhile we can use Lemma 2.8 to prove the following "cancellation" law.

Theorem 2.9 If $x + 1 = y + 1$, then $x = y$.

Proof $x + 1 = y + 1 \Rightarrow x \cup \{x\} = y \cup \{y\} \Rightarrow$ either $x = y$ or $x \in y$ and $y \in x$. The latter case is a contradiction to lemma 2.8. ■

Since the collection of all sets is an object that can be contemplated the study of collections can be extended to include collections that are not sets.

Collections that may or may not be sets are called classes. It is not within the scope of this book to study classes but we give the following definition.

Definition A collection that is not a set is called a **proper class**.

Exercises Prove the following "distributive" laws.

1. $a \cup \big(\bigcap_{\lambda \in \Lambda} b_\lambda \big) = \bigcap_{\lambda \in \Lambda} \big(a \cup b_\lambda \big)$
2. $a \cap \big(\bigcup_{\lambda \in \Lambda} b_\lambda \big) = \bigcup_{\lambda \in \Lambda} \big(a \cap b_\lambda \big)$

   Also show

3. $(a, b) = (c, d) \;\Rightarrow\; a = c$ and $b = d$.