Sections 8.1:
Limits of Sequences of Numbers

Definition A **sequence** is a function whose domain is the natural numbers $\mathbf{N}=\{1, 2, 3, \cdots\}$

We represent a sequence by $\{a_n\}$ where a_n is the image of n in the sequence. For example $\{2^n\}$ is the sequence $\{(1, 2), (2, 4), (3, 8), \cdots\}$. We commonly just list the elements of the image, so $\{2^n\} = \{2, 4, 8, \cdots\}$

Definition A sequence $\{a_n\}$ **converges** to the number L, if for every $\epsilon > 0$ there is a natural number N such that

$$n \geq N \Rightarrow |a_n - L| < \epsilon.$$

If no such number L exists we say that $\{a_n\}$ diverges.

If $\{a_n\}$ converges to L we write $\lim a_n = L$.

Example $\lim 1/n = 0$, since for any $\epsilon > 0$ pick $N > 1/\epsilon$, then if $n \geq N$ we have $|\frac{1}{n} - 0| < \epsilon$.

Theorem The limit laws for sequences.

If $\{a_n\}$ and $\{b_n\}$ are sequences of real numbers and A and B are real numbers, where $\lim a_n = A$ and $\lim b_n = B$, then the following rules hold:

1. Sum Rule: $\lim(a_n + b_n) = A + B$
2. Difference Rule: $\lim(a_n - b_n) = A - B$
3. Product Rule: $\lim(a_n \cdot b_n) = A \cdot B$
4. Constant Multiple Rule: $\lim(k \cdot b_n) = k \cdot B$
5. Quotient Rule: $\lim(\frac{a_n}{b_n}) = \frac{A}{B}$ if $B \neq 0$

Theorem The Sandwich Theorem for Sequences.

If $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are sequences of real numbers, and if $a_n \leq b_n \leq c_n$ for all $n \geq N$ for some natural number N, and if $\lim a_n = \lim c_n = L$, then $\lim b_n = L$.

Example Compute $\lim \frac{\sin n}{n}$.

Solution $\frac{-1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}$, and $\lim \frac{-1}{n} = \lim \frac{1}{n} = 0$, thus $\lim \frac{\sin n}{n} = 0$.

Theorem The Continuous Function Theorem for Sequences.

If $\{a_n\}$ is a sequence of real numbers, and if $\lim a_n = L$, and if f is a function that is continuous at L and defined at all a_n, then $\lim f(a_n) = f(L)$.

Theorem

If $f(x)$ is a function defined for all $x \geq N$ and $\{a_n\}$ is a sequence of real numbers such that $a_n = f(n)$ for all $n \geq N$, then $\lim\limits_{x\to\infty} f(x) = L \Rightarrow \lim a_n = L$.

Example Show that $\dfrac{\ln n}{n} = 0$.

Solution

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = \frac{0}{1} = 0.$$

Thus

$$\lim \frac{\ln n}{n} = 0.$$

Recommended Problems

pp 617-9, # 1, 3, 6, 10, 11, 13, 16, 17, 22, 23, 26, 30, 33, 39, 44, 53, 56, 62, 65, 72.