Sections 6.3:
Linear First-order Differential Equations

Definition A Linear First-Order Differential equation is a differential equation than can be written in the form

$$\frac{dy}{dx} + P(x) \cdot y = Q(x)$$

Example Write the differential equation $x\frac{dy}{dx} + 3y = \frac{\sin x}{x^2}$ in linear form.

Solution Dividing by x gives us $\frac{dy}{dx} + \frac{3}{x} \cdot y = \frac{\sin x}{x^3}$.

Theorem

The solution to the linear differential equation $\frac{dy}{dx} + P(x) \cdot y = Q(x)$ is

$$y = \frac{1}{v(x)} \int v(x) \cdot Q(x)\, dx$$

$$\text{where} \quad v(x) = e^{\int P(x)\, dx}.$$

The proof of this theorem is given on pages 504-5 in Thomas.

Example Solve the initial value problem $x\frac{dy}{dx} + 2y = x^3, x > 0$ and $y(2) = 1$.

Solution First we find the general solution. We Put the equation in standard form to get $\frac{dy}{dx} + \frac{2}{x} \cdot y = x^2$. Thus $P(x) = \frac{2}{x}$ and $Q(x) = x^2$, and we compute $v(x) = e^{\int \frac{2}{x}\, dx} = e^{2 \ln x} = x^2$. Thus the general solution is

$$y = \frac{\int x^2 \cdot x^2\, dx}{x^2} = \frac{1}{5}x^3 + C.$$

We now use the initial condition to find the specific solution.
$1 = y(2) = \frac{8}{5} + C \Rightarrow C = -\frac{3}{5}$. Thus the solution is $y = \frac{1}{5}x^3 - \frac{3}{5}$.

Example If you have \$1000 to open an account and you **continuously** add \$1000 per year, and the account earns 10% continuously compounded interest, then the number of dollars in your account at time t years will satisfy the initial value problem

$$\frac{dx}{dt} = 1000 + 0.10x, \quad x(0) = 1000.$$

1. Solve the initial value problem for x as a function of t.
2. How many years will it take for the amount in your account to reach \$1,000,000?

Solution We put the equation in standard form. $\frac{dx}{dt} - 0.10x = 1000$. Thus $P(t) = -0.10$, $Q(t) = 1000$ and $v(t) = e^{\int -.1\,dt} = e^{-.1t}$. Thus the general solution is

$$x(t) = \frac{1}{e^{-.1t}} \int 1000e^{-.1t}\,dt = 1000e^{.1t}\big(-10e^{-.1t} + C\big) = -10000 + 1000e^{.1t} \cdot C.$$

Using the initial condition that $x(0) = 1000$ we can solve for C.

$$1000 = x(0) = -10000 + 1000C \Rightarrow C = 11.$$

Thus we have

$$x(t) = 11000e^{.1t} - 10000.$$

Now when $x(t) = 1000000$ we have $1000 = 11e^{.1t} - 10 \Rightarrow t = 10 \ln \frac{1010}{10} \approx$ 45.2 years.

Recommended Problems

pp 510-1, # 1, 3, 8, 13, 16, 18, 23-26, 29.