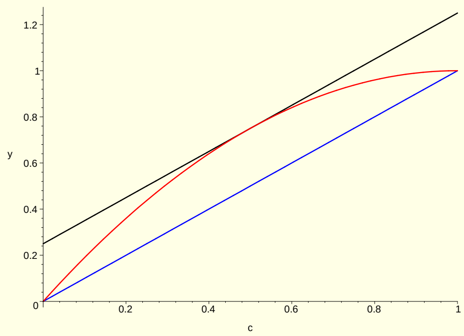
Sections 3.2:
The Mean Value Theorem and Differential Equations

In this section we examine an extremely important theorem known as the **Mean Value Theorem**. The importance of this theorem is most realized in its significance in developing the theory of calculus and other mathematics. It is an important tool in the argument of why certain statements are in fact true.

The mean value theorem says that if a function is differentiable, the average rate of change between two values is in fact equal to the instantaneous rate of change at some value between the two points for which the average rate of change is computed.

The Mean Value Theorem: If f is a continuous function on [a,b] and differentiable on (a,b), then $\exists\, c \in (a,b)$ such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$.



The proof of the Mean Value Theorem requires another theorem called Rolle's Theorem:

Rolle's Theorem: If f is a continuous function on $[a, b]$, differentiable on (a, b) and $f(a) = f(b)$, then $\exists\, c \in (a, b)$ such that $f'(c) = 0$.

A simplistic example of the use of the Mean Value Theorem is, if a person drives the 240 miles from Wibaux Montana to Billings Montana in 3 hours, her average speed is 80 mph. By the MVT there must be some point between Wibaux and Billings where she was driving 80 mph, and hence could receive a speeding ticket since the speed limit in Montana is now 75 mph.

A more important example is the following Theorem:

Theorem: If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on (a, b).

Proof: Assume there are values h and k in (a, b) where $f(h) \neq f(k)$, then by the MVT there must exist a value $c \in (h, k)$ where $f'(c) = \dfrac{f(k) - f(h)}{k - h} \neq 0$. This contradicts our hypothesis, hence there does not exist such values h and k. Thus f is constant.

This theorem has an important corollary.

Corollary: If $f'(x) = g'(x)$ for all $x \in (a, b)$, then $f(x) = g(x) + k$ for some constant k,

Proof: Let $h(x) = f(x) - g(x)$, then $h'(x) = f'(x) - g'(x) = 0$. Thus h is constant, so $f(x) - g(x) = k$, and finally $f(x) = g(x) + k$, which is what we wanted to show.

Definition: A **differential equation** is an equation relating an unknown function and one or more of its derivatives. A function whose derivatives satisfy a differential equation is called a **solution** to the differential equation.

Example: Find a function $f(x)$ where $f'(x) = \sin x$ and where $f(0) = 2$. (This is known as an initial value problem since we know the initial value of the function.)

Solution: by differentiating $C - \cos x$ we see that $f(x) = C - \cos x$ is a solution to the differential equation. Now $2 = f(0) = C - \cos 0 = C - 1$, thus $C = 3$. So our function is $f(x) = 3 - \cos x$.

Recommended Problems

pp 261-3, #2, 4, 6, 10, 14, 18, 20, 22, 28, 30, 32, 36, 38, 40, 42.